BỘ GIÁO DỤC VÀ ĐÀO TẠO

TRƯỜNG ĐẠI HỌC MỎ-ĐỊA CHẤT

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FUNDAMENTALS OF HERMITIAN AND KAHLERIAN GEOMETRY

Th.S Nguyễn Thị Kim Sơn

Hà nội, tháng 5 năm 2020

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Fundamentals of Hermitian and Kählerian Geometry

1. Almost Complex Structure

Let V be a vector space over the field $\mathbb{R}$ of real numbers. Assume that V admits a linear map $J : V \to V$ satisfying $J^2 = J \circ J = -I$ (where I represents the identity map). It is an exercise to show that $\dim V$ must be even in order for such a J to exist.

Such a J is called an *almost complex structure* on V and the vector space V equipped with J is called an *almost complex vector space.*

Now, consider[1] the complexification $V^{\mathbb{C}} := \mathbb{C} \otimes V$. The complex vector space $V^{\mathbb{C}}$ is of complex dimension $2m$. J extends to a complex linear map, with $J^2 = -I$.

The linear map J has only 2 eigenvalues $\pm i$. Consider the respective eigenspaces:

$$V' := \{v \in V^{\mathbb{C}} \mid Jv = iv\} \quad \text{and} \quad V'' := \{v \in V^{\mathbb{C}} \mid Jv = -iv\}.$$

Obviously, $V' \oplus V'' = V^{\mathbb{C}}$, and $\dim_{\mathbb{C}} V' = m = \dim_{\mathbb{C}} V''$. It is easy to verify that

$$V' = \{u - iJu \mid u \in V\} \quad \text{and} \quad V'' = \{u + iJu \mid u \in V\}.$$

2. Tangent Space and Bundle

Let M be a complex manifold of dimension m. Then it is also a smooth manifold. Let $p \in M$ and let T_pM be its tangent space, which is a vector space of dimension $2m$. Let TM denote the tangent bundle given by $TM = \bigcup_{p \in M} T_pM$, as usual in the manifold theory.

Since M is a complex manifold, it comes with the natural almost complex structure J, which we are going to describe now. We shall do it in terms of coordinates. Take a coordinate system $(z_1, \ldots, z_m) : U \to \mathbb{C}^m$ from a coordinate neighborhood U about $p \in M$. Write $z_k = x_k + iy_k$ for each k. Notice that the vectors

$$\left.\frac{\partial}{\partial x_1}\right|_p, \left.\frac{\partial}{\partial y_1}\right|_p, \quad \cdots \quad , \left.\frac{\partial}{\partial x_m}\right|_p, \left.\frac{\partial}{\partial y_m}\right|_p$$

span the real tangent space T_pM. Define $J_p : T_pM \to T_pM$ by

$$J_p\left(\left.\frac{\partial}{\partial x_k}\right|_p\right) = \left.\frac{\partial}{\partial y_k}\right|_p, J_p\left(\left.\frac{\partial}{\partial y_k}\right|_p\right) = -\left.\frac{\partial}{\partial x_k}\right|_p$$

for each $k = 1, 2, \ldots, m$ and extend it linearly over $\mathbb{R}$. Then $p \in M \mapsto J_p \in (T_pM)^* \otimes T_pM$ is a smooth map. Hence this correspondence shows that J is a smooth section of the bundle $T^*M \otimes TM$. This is an almost complex structure of M.

Now, we shall complexify T_pM, and consequently TM. We do this by extending coefficients. Namely, we let

$$\mathbb{C}T_pM := \mathbb{C} \otimes T_pM \quad \text{and} \quad \mathbb{C}TM := \mathbb{C} \otimes TM.$$

In local coordinates, the complexification simply means allowing complex values for coefficients for the real tangent vectors and tangent vector fields.

Extend J to the complex tangent spaces and bundles $\mathbb{C}$-linearly, following the formalism introduced above. Then consider the respective eigenspaces of J_p. They are

$$T'_pM = \{u - iJu \mid u \in T_pM\} \quad \text{and} \quad T''_pM = \{u + iJu \mid u \in T_pM\}.$$

Traditional notation in local complex coordinates is worth mentioning at this juncture. They appear quite naturally now:

$$\frac{\partial}{\partial z_k} = \frac{1}{2}\left(\frac{\partial}{\partial x_k} - iJ\left(\frac{\partial}{\partial x_k}\right)\right) = \frac{1}{2}\left(\frac{\partial}{\partial x_k} - i\,\frac{\partial}{\partial y_k}\right)$$

and

$$\frac{\partial}{\partial \bar{z}_k} = \frac{1}{2}\left(\frac{\partial}{\partial x_k} + i\,\frac{\partial}{\partial y_k}\right),$$

where the factor $\frac{1}{2}$ is introduced for reasons one will soon see.

Notice that the Cauchy-Riemann equations for a mapping $f\colon M \to N$ between two complex manifolds M and N are equivalent to the equation $J_N \circ df = df \circ J_M$, where J_M, J_N are the almost complex structures constructed for M, N respectively.

One sees also that there is a natural $\mathbb{R}$-linear isomorphism (identification) between T'_pM and T_pM defined by

$$v \in T'_pM \mapsto \text{Re } v \in T_pM.$$

Notice, however, that T'_pM is a complex vector space of complex dimension m, whereas T_pM is a real $2m$ dimensional space with no prescribed complex vector space structure.

Altogether, we have introduced four tangent spaces $T_pM, \mathbb{C}T_pM, T'_pM$ and T''_pM. They appear naturally for a complex manifold M, and of course they give rise to respective bundles.

3. Cotangent Space and Bundle

For the cotangent spaces and bundles, we shall simply build upon what we developed with the tangent spaces and bundles. The set of all $\mathbb{C}$-linear functionals on $\mathbb{C}T_pM$ will be the space we work in. With the basis

$$\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_m}; \frac{\partial}{\partial \bar{z}_1}, \ldots, \frac{\partial}{\partial \bar{z}_m}$$

we shall take its dual basis. One can quickly check that the dual basis consist of complex co-vectors at p given by

$$dz_k := dx_k + idy_k, \qquad d\bar{z}_k := dx_k - idy_k,$$

for $k = 1, \ldots, m$. (This is the reason for $\frac{1}{2}$ in the previous section because we customarily want $dz_j(\partial/\partial z_j) = 1$ and so forth.) Likewise one sees that

$T_p^{1,0}M := (T_p'M)^*$ is the vector space over $\mathbb{C}$ generated by $dz_1|_p, \ldots, dz_m|_p$, and that $T_p^{0,1}M := (T_p''M)^*$ by $d\bar{z}_1|_p, \ldots, d\bar{z}_m|_p$.

It may be a good practice for the sake of symbolic calculus, to verify the notational reasonability such as

$$df = \sum_{j=1}^{m} \frac{\partial f}{\partial z_j} dz_j + \sum_{j=1}^{m} \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j$$

for any smooth function $f : M \to \mathbb{C}$. Likewise one may define and develop the concept of complex differential forms of bi-degree (p, q) and their tensor products. However we shall not provide any further details.

4. Connection and Curvature

We now introduce the connections and curvatures briefly.

4.1. Riemannian connection and curvature

Let $\mathfrak{X}(M)$ denote the set of smooth vector fields on M.

Definition 4.1. A *linear connection* on the tangent bundle TM over the manifold M is a map $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M) : (X, Y) \mapsto \nabla_X Y$ satisfying:

(1) $\nabla_{f_1 X_1 + f_2 X_2} Y = f_1 \nabla_{X_1} Y + f_2 \nabla_{X_2} Y$ for any $f_1, f_2 \in \mathcal{C}^\infty(M)$ and any $X_1, X_2, Y \in \mathfrak{X}(M)$.
(2) $\nabla_X (aY_1 + bY_2) = a\nabla_X Y_1 + b\nabla_X Y_2$ for any $a, b \in \mathbb{R}$ and any $X, Y_1, Y_2 \in \mathfrak{X}(M)$.
(3) $\nabla_X (fY) = f\nabla_X Y + (Xf)Y$, for any $f \in \mathcal{C}^\infty(M)$ and any $X, Y \in \mathfrak{X}(M)$.

Linear connections are also called *affine connections*. For a differentiable manifold, there are infinitely many such connections. On the other hand, each such connection provides a method of differentiating a smooth vector field by another. Thus the linear connection is in fact a “differentiation”.

Of course it is natural to look for a connection that can explain the particular geometry one aims to study. In our case that is the complex geometry, which concerns quantities such as the (almost) complex structure J and the Hermitian metric just introduced.

If we discount the complex structure concentrate on the metric structure (and consequently our manifold is just Riemannian), the natural and well-known connection is the *Levi-Civita connection* (i.e., the *Riemannian covariant differentiation*). Since the (real) Hermitian metric is Riemannian, we shall start with the Levi-Civita connection.

Definition 4.2. Let (M, h) be a Riemannian manifold. (The Hermitian metric h is also a real Riemannian metric.) Then the *Levi-Civita connection* on (M, h) is a linear connection ∇ satisfying the following two additional conditions:

$$(4)\quad \tau(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y] = 0$$
$$(5)\quad (\nabla h)(X, Y, Z) := X(h(Y, Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z) = 0,$$

where the notation $[X, Y]$ stands for the Lie bracket of two vector fields X, Y.

It is well-known that the Levi-Civita connection exists and is unique (cf. [Greene 1987], [Kobayashi and Nomizu 1969], e.g.). The quantity τ is called the torsion tensor, and thus the (4) is called the torsion-free condition. (5) is commonly referred to as the condition that the metric is parallel. Of course this Levi-Civita connection is the key concept toward Riemannian geometry. It determines the geodesics, parallelism and the curvature.

4.2. Riemann curvature tensor and sectional curvature

Now we are ready to introduce the Riemannian curvature(s). In case the manifold is real two dimensional, the curvature is a function. However in higher dimensional case, the curvature is a multi-linear form on vector fields.

Let (M, J, h, ∇) be a complex manifold with a Hermitian metric h and its Levi-Civita connection ∇. We start with the (Riemannian) sectional curvature. Let $X, Y, Z, W \in \mathfrak{X}(M)$. Then we define the following notation:

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

$$R(X,Y,Z,W) = h(R(X,Y)Z, W).$$

Note that the last is a real-valued function, 4-linear on $\mathcal{C}^\infty(M)$. It is "point-wise" meaning that the value $R(X,Y,Z,W)\big|_p$ of $R(X,Y,Z,W)$ at $p \in M$ depends only on the point-values at p of the vector fields X, Y, Z and W.

Since this full curvature tensor is hard to use in general, one often considers the concept called the *Riemannian sectional curvature*. To define this, consider $X, Y \in \mathfrak{X}(M)$ that are linearly independent at $p \in M$ over $\mathbb{R}$. Then the value

$$K_p(X,Y) := -\frac{R(X,Y,X,Y)}{\|X \wedge Y\|^2}\ \Big|_p$$

is the sectional curvature at p along the 2-dimensional plane in T_pM generated by X_p and Y_p, where $\|X \wedge Y\|^2 = h(X,X)h(Y,Y) - h(X,Y)^2$. It is not hard to check that this value of the sectional curvature depends only on the 2-dimensional plane (i.e., section) spanned by X_p and Y_p, but not on the choice of the basis vectors X_p and Y_p. In case the manifold is a real 2-dimensional surface in $\mathbb{R}^3$ equipped with the induced metric, that is its first fundamental form, then this sectional curvature coincides with the Gauss curvature.

4.3. Holomorphic sectional curvature

Now we re-instate the complex structure J back into consideration. Thus our manifold is now Hermitian. At this stage we have to re-consider our choice for the connection. Namely we have to consider which properties we would like to have for our linear connection to satisfy. Decision must be made among the following three properties:

(P1) $(\nabla h)(X,Y,Z) := X(h(Y,Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z) = 0.$
(P2) Torsion-free, i.e., $\tau(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y] = 0.$
(P3) $(\nabla J)(X,Y) := \nabla_X(J(Y)) - J(\nabla_X Y) = 0.$

It is known that all three can be satisfied only if the metric h is special. Such a metric is called *Kählerian* (or simply *Kähler*). Several necessary and sufficient conditions for the metric to be Kähler are known as follows:

Proposition 4.1. *For a complex manifold M with the complex Hermitian metric h', consider a complex local coordinate system $(z_1, \ldots, z_n)$, and let $h'_{j\bar{k}} = h'\Big(\dfrac{\partial}{\partial z_j}, \dfrac{\partial}{\partial z_j}\Big)$ and $\omega = \sum h_{j\bar{k}} dz_j \wedge d\bar{z}_k$. Then the following are equivalent:*

(i) h (or, equivalently, its complex form h') is Kähler, i.e., the Levi-Civita connection ∇ with respect to the metric h satisfies $\nabla J = 0$.
(ii) $d\omega = 0$.
(iii) There exists a smooth function φ such that $h'_{j\bar{k}} = \dfrac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_k}$.

Many well-known metrics are Kähler: the Poincaré metric on the disc and the Bergman metric of bounded domains in $\mathbb{C}^n$ are good examples.

On the other hand, general Hermitian metrics are not Kähler. In such a case what connection should be taken? It is generally agreed that condition (P3) $\nabla J = 0$ should be taken, but the "torsion-free" condition (P2) is dropped, allowing the *torsion tensor* τ in (P2) to be non-zero.

Regardless, when the manifold is Hermitian, one can make sense of "holomorphic sections"—those 2-dimensional plane in T_pM spanned by X_p and JX_p for some non-zero $X_p \in T_pM$ and the (Riemann) sectional curvature along this plane. Of course two vectors are linearly independent over $\mathbb{R}$ as we see from $h_p(X_p, JX_p) = 0$. Thus the *holomorphic sectional curvature* in the direction of X at p is defined to be $K_p(X, JX)$. (In Kählerian case, the holomorphic sectional curvature is indeed the Riemann sectional curvature for a holomorphic section.)

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5. Connection and Curvature in Moving Frames

5.1. Hermitian metric, frame and coframe

Even though we deal mostly with Kählerian case (where the torsion tensor τ vanishes), it is going to be useful for the future developments to introduce the general Hermitian case.

Let $T'M$ represent the holomorphic tangent bundle. Given an Hermitian metric, it is possible to choose a smoothly varying orthonormal basis (usually called a *unitary frame*)

$$e_1, \ldots, e_m$$

in a local coordinate neighborhood. This can be done, for example, by applying the Hermitian Gram-Schmidt process to the coordinate frame $\frac{\partial}{\partial z_1}, \cdots, \frac{\partial}{\partial z_m}$. (Note that the unitary frame therefore is smooth, but not consisting of holomorphic vector fields in general.)

Then consider its dual, that is the (holomorphic) cotangent bundle $T^{1,0}M$, whose sections are called the (smooth) $(1,0)$-forms. Take the basis for sections of $T^{1,0}M$ dual to the frame chosen above and denote it by

$$\theta_1, \ldots, \theta_m.$$

This particular basis is called a *unitary coframe.*

Then the Hermitian metric can be written by

$$ds^2 = \sum_{i=1}^{m} \theta_i \otimes \bar{\theta}_i.$$

5.2. Hermitian connection

We now introduce the connection we shall use, continuing the discussion of the preceding section (with the same notation). We feel however that this part of exposition can be quite terse—thus we give an example here which illustrates how a connection can be interpreted in terms of a certain matrix of 1-forms. The reader may skip this example if they are familiar with such matters.

Example 4.1. Let (M, g) be a Riemannian manifold and let ∇ be the Levi-Civita connection. Take a local coordinate neighborhood and a local coordinate system $x_1, \ldots, x_m$. Let

$$e_j = \frac{\partial}{\partial x_j},$$

for $j = 1, \ldots, m$. Then it is customary to write

$$\nabla_{e_i} e_j = \sum_k \Gamma_{ij}^k e_k.$$

The functions Γ_{ij}^k are the (2nd) Christoffel symbols. The Leibniz rule which the connection ∇ satisfies is

$$\nabla_{e_i}(\psi e_j) = e_i(\psi) \cdot e_j + \psi \cdot \sum_k \Gamma_{ij}^k e_k.$$

Now, considering the meaning of the differential forms and the sections of bundles involved, one can now makes sense of the expression:

$$\nabla \colon \Gamma(TM) \to \Gamma(TM \otimes T^*M)$$

given by

$$\nabla\Big(\sum_{j=1}^{m}\psi^j e_j\Big)=\sum_{j=1}^{m}\Big((d\psi^j)\otimes e_j+\sum_{k=1}^{m}\psi^k\theta_{kj}\otimes e_j\Big).$$

The relation between the connection form (a matrix, in fact, of 1-forms) and the Levi-Civita connection ∇ should be visible from this, at least. (Of course this does not explain fully how all the other properties (such as torsion (free) condition, metric compatibitity etc.) of connection matrix and related concepts (such as curvature and others) are developed and computed. For further information, cf., e.g., [Chern 1979] and [Chern 1968]).

We return to the Hermitian case and choose a suitable connection form on the m-dimensional Hermitian manifold M. Cartan's method says[2] that the connection matrix can be chosen from the following equation

$$d\theta_i=\sum_{j=1}^{m}\theta_j\wedge\theta_{ji}+\tau_i.$$

Notice that neither θ_{ji} nor τ_i are determined through this identity. Hence there are (infinitely) many choices for the connection form θ_{ji} and the torsion form τ_i. Rather, one needs to put extra assumptions in order to select the suitable connection matrix (as well as the torsion). A good example, which we use is the *canonical Hermitian connection* (i.e., the Chern connection), which is the choice of θ_{ji} satisfying the conditions:

$$\theta_{ij}+\overline{\theta_{ji}}=0$$

and

$$\tau_i=\frac{1}{2}\sum_{j,k=1}^{m}T_{ijk}\theta_j\wedge\theta_k.$$

Note that this last requires that the torsion is of type $(2,0)$ only. (No $(1,1)$ part exists. And, of course, the whole τ vanishes in the Kähler case.)

5.3. Curvature

The *curvature form* is defined to be

$$\Theta_{ij} = d\theta_{ij} - \sum_{k=1}^{m} \theta_{ik} \wedge \theta_{kj}.$$

One may check that the identity $\Theta_{ij} = -\overline{\Theta}_{ji}$ holds for the curvature form. Also,

$$\Theta_{ij} = \frac{1}{2} \sum_{k,\ell=1}^{m} R_{ijk\ell} \theta_k \wedge \overline{\theta_\ell}.$$

Namely, the curvature form Θ_{ij} are of type $(1,1)$. Notice that the skew-Hermitian symmetry for the curvature form above is equivalent to

$$R_{ijk\ell} = \overline{R_{ji\ell k}}.$$

In this notation, the holomorphic sectional curvature, the bisectional curvature and the Ricci curvature are easy to define. They are, respectively,

- The *holomorphic sectional curvature* in the direction of vector field $\eta = \sum_{k=1}^{m} \eta_k e_k$ is

$$\frac{\sum_{i,j,k,\ell=1}^{m} R_{ijk\ell} \eta_i \bar{\eta}_j \eta_k \bar{\eta}_\ell}{\left(\sum_{i=1}^{m} \eta_i \bar{\eta}_i\right)^2}.$$

- The (*holomorphic*) *bisectional curvature* determined by $\xi = \sum_{k=1}^{m} \xi_k e_k$ and $\eta = \sum_{k=1}^{m} \eta_k e_k$ is

$$\frac{\sum_{i,j,k,\ell=1}^{m} R_{ijk\ell} \xi_i \bar{\xi}_j \eta_k \bar{\eta}_\ell}{\left(\sum_{i=1}^{m} \xi_i \bar{\xi}_i\right) \left(\sum_{i=1}^{m} \eta_i \bar{\eta}_i\right)}.$$

- The *Ricci tensor* is given by

$$R_{ij} = \sum_{k=1}^{m} R_{ijkk},$$

and

$$\mathrm{Ric}(\xi, \eta) = \sum_{i,j=1}^{m} R_{ij} \xi_i \bar{\eta}_j.$$

5.4. The Hessian and Laplacian

For a smooth function $u : M \to \mathbb{R}$ on the Riemannian manifold M, the *Hessian* of u is the *second covariant derivative* that is defined[3] to be, in the Riemannian covariant derivative notation,

$$Hess(u)(X, Y) = \nabla^2 u(X, Y) := X(Yu) - (\nabla_X Y)u$$

for every $X, Y \in \mathfrak{X}(M)$. The *Laplacian* Δu of u is defined as the *trace of* $Hess(u)$.

For Hermitian manifold M of real dimension $2m$, let $e_1, \ldots, e_{2m}$ be a real-orthonormal basis of T_pM. Then

$$\Delta u(p) = \sum_{i=1}^{2m} Hess\Big|_p (u)(e_i, e_i).$$

For the same Hermitian manifold M, the *complex Laplacian* of u, is defined using moving frame approach as follows: one writes

$$du = \sum_{i=1}^{m} u_i \theta_i + \sum_{i=1}^{m} \bar{u}_i \bar{\theta}_i.$$

Taking one more exterior derivative (with connection forms) one can define u'_{ij}, u_{ij} by

$$du_i - \sum_j u_j \theta_{ij} = \sum_j u'_{ij} \theta_j + u_{ij} \bar{\theta}_j.$$

Define the *complex Laplacian* of u by

$$\Delta_c\, u = \sum_i u_{ii}.$$

Remark 4.2. It is important to realize that the Laplacian of a function is the trace of its second covariant differentiation. Notice therefore that the Laplacian Δ_c above relies upon the canonical Hermitian connection ∇.