BỘ GIÁO DỤC VÀ ĐÀO TẠO
TRƯỜNG ĐẠI HỌC MỎ-ĐỊA CHẤT

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TWO APPLICATION OF THE SCHWARZ LEMMA

Th.S Nguyễn Thị Kim Sơn

Hà nội, tháng 5 năm 2020

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CHAPTER 1. ABSTRACT

We exhibit two applications of Schwarz lemmas in several complex variables. The first application extends Fornæss and Stout's theorem on monotone unions of balls to monotone unions of ellipsoids. The second application extends Yang's theorem on bidiscs to the generalized bidisc defined by the author in his previous work. These applications reveal a connection between the geometry of domains and their curvatures. The proof contains a careful study of biholomorphisms, a detailed analysis on convergences, and a modified argument of Yang.

The most striking and influential result in complex analysis of one variable is the Riemann mapping theorem. It asserts that all proper simply connected open subsets in C are biholomorphic onto the unit disc. Thus, it was hoped that a similar theorem could be proved in C^n for higher dimensions $n > 1$. In 1960, Poincaré showed the bidisc $D^2 = \{(z, w) : |z| < 1 \text{ and } |w| < 1\}$ is not biholomorphic to the ball $B^2 = \{(z, w) : |z|^2 + |w|^2 < 1\}$. This negated the expectation and motivated a new study on biholomorphism in several complex variables.

On the other hand, Fornæss and Stout [1977] showed that a Kobayashi hyperbolic manifold M is biholomorphic onto the unit ball B^m, provided that M admits a monotone union of B^m. Their theorem gives a version of the Riemann mapping theorem in high dimensions under some circumstances. In this paper, we follow this fashion and exhibit a theorem about monotone unions of ellipsoids

$E_n := \{(z, w) \in \mathbb{C}^2 : |z|^2 + |\omega|^{2n} < 1\}$. More precisely, we obtain the following theorem.

Theorem 1. *Let M be a two-dimensional Hermitian manifold with a real bisectional curvature bounded from above by a negative -K, and assume M is a monotone union of ellipsoids E_n for some $n \in \mathbb{Z}^+$. Then M is either biholomorphic onto E_n or onto the unit ball* B^2.

This theorem generalizes Fornæss and Stout's theorem on monotone unions of balls to monotone unions of ellipsoids in dimension 2. We remark that Fornæss and Stout's original proof is hard to be adapted into our theorem. Among other difficulties, the situation that biholomorphisms converge to a constant map is hard to be excluded. This difficulty is easy to be resolved in Fornæss and Stout's theorem because of symmetries of balls. However, the shape of ellipsoids is more irregular than balls. Hence, in order to resolve this difficulty we make local estimates around accumulation points and use the estimates to reconstruct biholomorphisms. This new technique is exhibited in Section 2.

The readers are reminded that this theorem does not belong to a classical topic on automorphism groups. For the classical topics on automorphism groups, readers are referred to [Bedford and Pinchuk 1991; 1998; Greene and Krantz 1991; 1993; Wong 1977].

The other application of Schwarz lemmas in this article is about curvature bounds. In the 1970s, Yang [1976] showed that on polydiscs, there do not exist complete Kähler metrics with bounded holomorphic bisectional curvatures. Yang's discovery was recently generalized to product manifolds by Seshadri and Zheng [2008] and Seo [2012]. On the other hand, the author introduced a new type of domains called the generalized bidiscs in [Liu 2017]. It is known that some generalized bidiscs are biholomorphic to bounded domains in C^2. The generalized bidiscs are defined to be $D \times H^+ := \{(z, w) : z \in D \text{ and } \omega \in e^{i\theta(z)} H^+\}$. Here D denotes the unit disc, H^+ denotes the upper half plane, θ denotes a continuous real function depending on z, and $e^{i\theta(z)}H^+$ denotes the upper half plane rotated by the angle θ *(z)*. The generalized bidiscs are, in general, not product manifolds. However, in this paper, we show they share similar geometric properties with bidiscs. That is, some generalized bidiscs do not admit complete Kähler metrics with bounded negative holomorphic bisectional curvatures. More precisely, we show:

Theorem 2. *Let* $k \in (0,\pi)$ *and* $\theta(z) \in [0,k)$ *for all* $z \in \mathrm{D}$. *Then there do not exist two numbers* $d > c > 0$ *and a complete Kähler metric on* $\mathrm{D} \times \mathrm{H}^+$ *such that the holomorphic bisectional curvature is between —d and —c.*

These results about curvature bounds are discussed in Section 3.

Let $n \in \mathbb{Z}^+$. It is classical to define ellipsoids $E_n \subset \mathbb{C}^2$ by

$E_n = \{(z, w) : |z|^2 + |w|^{2n} < 1\}$.

Let M be a manifold with dimension m. In this paper, we say M is a monotone union of ellipsoids E_n via f_j if

(1) there exists a sequence of open subsets $M_j \subset M$ so that $M_j \subseteq M_{j+1}$,

(2) each M_j is biholomorphic, by f_j, onto the ellipsoids E_n, and

(3) $M = \bigcup_j M_j$.

Remark 1.1. We sometimes omit "via f_j" and only say "M is a monotone union of ellipsoids E_n".

Remark 1.2. Similarly, one can define a monotone union of for an arbitrary domain .

We also recall some terminologies on Kähler manifolds. Let (M, J, h) be a Kähler manifold M of dimension m with a Kähler metric h and a complex structure J. The curvature tensor R on (M, J, h) is given by

$$R_{i\bar{j}k\bar{l}} = \frac{\partial^2 h_{i\bar{j}}}{\partial z_k \partial \bar{z}_l} - \sum_{\alpha,\beta=1}^{m} h^{\alpha\bar{\beta}} \frac{\partial h_{i\bar{\beta}}}{\partial z_k} \frac{\partial h_{\alpha\bar{j}}}{\partial \bar{z}_l}$$

in local coordinates $(z_1, \ldots, z_n)$. The holomorphic bisectional curvature for $X \in T_pM$ at $p \in M$ is given by

$$B(X, Y) = -\frac{\sum_{i,j,k,l=1}^{m} R_{i\bar{j}k\bar{l}} X_i \overline{X}_j Y_k \overline{Y}_l}{\left(\sum_{i,j=1}^{m} h_{i\bar{j}} X_i \overline{X}_j\right)\left(\sum_{i,j=1}^{m} h_{i\bar{j}} Y_i \overline{Y}_j\right)},$$

where

$$X = \sum_{j=1}^{m} X_j \frac{\partial}{\partial z_j} + \sum_{j=1}^{m} \overline{X}_j \frac{\partial}{\partial \bar{z}_j}, \qquad Y = \sum_{j=1}^{m} Y_j \frac{\partial}{\partial z_j} + \sum_{j=1}^{m} \overline{Y}_j \frac{\partial}{\partial \bar{z}_j}.$$

We are going to remind readers with backgrounds on Schwarz lemmas and almost maximal principles.

Theorem 1.3 (the Schwarz lemma of [Yau 1978]). *Let $f : M \to N$ be a holomorphic mapping from a complete Kähler manifold (M, g) with its Ricci curvature bounded from below by a negative constant $-k$ into a Hermitian manifold (N, h) with its holomorphic bisectional curvature bounded from above by a negative constant $-K$. Then*

$$f^*h \le \frac{k}{K} g.$$

Theorem 1.4 (the almost maximal principles of [Yau 1978]; see also [Kim and Lee 2011]). *Let M be a complete Riemannian manifold M with the Ricci curvature bounded from below. Then for any C^2 smooth function $f : M \to \mathbb{R}$ that is bounded from above, there exists a sequence $\{p_k\}$ such that*

$$\lim_{k\to\infty} |\nabla T(p_k)| = 0, \qquad \limsup_{k\to\infty} \Delta T(p_k) \le 0, \qquad \lim_{k\to\infty} T(p_k) = \sup_M T.$$

Recently Yang and Zheng [2016] defined a new notion of curvatures on Hermitian manifolds called real bisectional curvature. We will briefly give the definition and discuss a version of Schwarz lemma in terms of real bisectional curvature as follows.

Definition 1.5 [Yang and Zheng 2016]. Let (M^n, g) be a Hermitian manifold, and denote by R the curvature tensor of the Chern connection. We say the real bisectional curvature of M is bounded from above by a constant C if

$$\sum_{i,j,k,l} R_{i\bar{j}k\bar{l}} \xi_{ij} \xi_{kl} \le C \operatorname{tr}(\xi^2),$$

for all nontrivial, nonnegative, Hermitian $n \times n$ matrices ξ.

Observe Theorem 4.5 in [Yang and Zheng 2016], and use the identity $\Delta v = 2\Box v$ for Kähler manifolds (here Δ is the regular Laplacian, $\Box$ is the complex Laplacian, and v is an arbitrary smooth function). One can easily obtain the following Schwarz lemma as a corollary of Theorem 4.5 in [Yang and Zheng 2016].

Theorem 1.6 (the Schwarz lemma of [Yang and Zheng 2016]). *Let $f : M \to N$ be a holomorphic mapping from a complete Kähler manifold (M, g) with its Ricci curvature bounded from below by a negative constant $-k$ into a Hermitian manifold (N, h) with its real bisectional curvature bounded from above by a negative constant $-K$. If ν is the maximal rank of the map f, then*

$$f^*h \le \frac{k\nu}{K} g.$$

CHAPTER 4. MONOTONE UNIONS OF ELLIPSOIDS

We discuss monotone unions of ellipsoids $E_n := \{(z, w) : |z|^2 + |w|^{2n} < 1\}$ in $\mathbb{C}^2$ in this section.

Let M be an m-dimensional complex manifold which is a monotone union of Ω via f_j. Take an arbitrary point $q \in M$, and let $j \to \infty$; then $\{f_j(q)\}_{j=1}^{\infty}$ has a limit point, possibly after passing to a subsequence, because of the boundedness of Ω. Then the location of limit point of $\{f_j(q)\}_{j=1}^{\infty}$ has two possibilities. The limit point of $\{f_j(q)\}_{j=1}^{\infty}$ can be either an interior point of Ω or a boundary point at $\partial\Omega$.

The following lemma settles the case that the limit of $f_j(q)$ is an interior point of Ω. From now on, we will not distinguish between the convergence of sequences and the convergence after passing to subsequences.

Lemma 2.1. *Let M be a m-dimensional Hermitian manifold with a real bisectional curvature bounded from above by a negative number $-K$. Assume M is a monotone union of $\Omega \subset \mathbb{C}^m$ via f_j where Ω is a bounded domain in $\mathbb{C}^m$ with a complete Kähler metric of which the Ricci curvature is bounded from below by a negative number $-k$. We also assume there exists an interior point $q \in M$ so that $f_j(q) \to p \in \Omega$. Then M is biholomorphic onto Ω.*

Proof. Since Ω is bounded, f_j is a normal family of biholomorphisms. Let f_j converge to a holomorphic map F. Considering the inverses $\{f_j^{-1}\}_{j=1}^{\infty}$, we want to show they are locally bounded in a small geodesic ball B_p centered $p \in \Omega$ with radius $\epsilon > 0$. Let d_M be the Hermitian metric of M and d_Ω be the Kähler metric of Ω. Indeed, by Theorem 1.6,

$$(f_j^{-1})^* d_M \leq C d_\Omega$$

for each $j > 0$, where $C = km/K$. Let $N > 0$ be so that $f_j(q) \in B_p$ for all $j > N$. Considering arbitrarily $w \in B_p$, we have

$$d_M(q, f_j^{-1}(w)) \leq C d_\Omega(f_j(q), w) < 2C\epsilon, \tag{1}$$

for $j > N$. This means f_j^{-1} is locally bounded (hence a normal sequence) in B_p. We denote the limit of $\{f_j^{-1}\}_{j=1}^{\infty}$ by G. One can see that $F \circ G(w) = w$ in B_p because $\{f_j\}_{j=1}^{\infty}$ is uniformly convergent on compact subsets of M and $\{f_j^{-1}\}_{j=1}^{\infty}$ is uniformly convergent on compact subsets of B_p.

More generally, $\{f_j^{-1}\}_{j=1}^{\infty}$ is locally bounded on Ω. Indeed, we consider two interior points $w', w'' \in M$ and use Theorem 1.6 for f_j^{-1} again:

$$d_M(f_j^{-1}(w'), f_j^{-1}(w'')) \leq C d_\Omega(w', w''). \tag{2}$$

From this, we can see that $f_j \circ G$ is well defined everywhere in Ω. Hence, $F \circ G$ is well defined on Ω. Since $F \circ G(w) = w$ for $w \in B_p$, and $F \circ G$ is a holomorphic map, by the identity theorem, we obtain that $F \circ G(w) = w$ for all $w \in \Omega$. This implies F is surjective.

Since $f_j(q) \to p \in \Omega$ as $j \to \infty$, it follows that $\det Jf_j(q) \not\to 0$ by Cartan's theorem. We claim the limit of $(\det Jf_j)(z)$ is nowhere vanishing for arbitrary $z \in M$, where J denotes the Jacobian. The reason is as follows. By the Cauchy estimates, the fact that $\{f_j\}_{j=1}^{\infty}$ is normal implies that $\{\det Jf_j\}_{j=1}^{\infty}$ is also normal. But $\{\det Jf_j\}_{j=1}^{\infty}$ is nowhere zero for each $j > 0$ because f_j is a biholomorphism and then by Hurwitz's theorem, $\det JF$ is a zero function or nowhere zero. And the claim follows by the fact that $\det Jf_j(q) \not\to 0$. Now $\det Jf_j(z) \not\to 0$ for all $z \in M$, and hence, $\det JF(z)$ is nonzero everywhere. This also implies $F(M)$ is open by the open mapping theorem.

We are going to show F is 1-1. For this, we consider two interior points $z', z'' \in M$ and use Theorem 1.6 for f_j^{-1} for each $j > 0$ again:

$$d_M(z', z'') \leq C d_\Omega(f_j(z'), f_j(z'')). \tag{3}$$

Since $\det Jf_j(z)$ does not approach zero for all $z \in M$, $f(z)$ does not approach the boundary $\partial\Omega$ for fixed $z \in M$. In particular, $f_j(z')$ and $f_j(z'')$ do not approach the boundary $\partial\Omega$ where the Kähler metric d_Ω blows up. From this, $F(z') = F(z'')$ implies $d_\Omega(f_j(z'), f_j(z'')) \to 0$. By (3), we obtain that $z' = z''$. Consequently, F is 1-1.

Hence, M is biholomorphic onto Ω via bijective F. □

By a similar argument, we can verify the following corollary. Instead of looking at only the exhaustive subsets of M in the previous lemma, the following corollary considers both exhaustive subsets of M and Ω.

Corollary 2.2. *Let M be an m-dimensional Hermitian manifold with a real bisectional curvature bounded from above by a negative number $-K$. Assume $M = \bigcup_j M_j$ where $M_j \subset M_{j+1}$ and f_j is a biholomorphism from M_j onto $\Omega_j \subset \Omega \subset \mathbb{C}^m$. Suppose Ω is a bounded domain in $\mathbb{C}^m$ and Ω_j is a complete Kähler manifold with the Ricci curvatures bounded from below by a same negative number $-k$ (independent with j). We also assume there exists a point $q \in M$ so that* $\det Jf_j(q) \not\to 0$. *Then F is* 1-1, *and hence, M is taut.*

For the sake of completeness, we also include a short outline of the proof.

Outline of proof. Since Ω is bounded, $\Omega_j \subset \Omega$ is bounded too for each $j > 0$. Hence, $\{f_j\}_{j=1}^{\infty}$ is still normal. By $\det Jf_j(q) \not\to 0$, we can see $\det Jf_j(z) \not\to 0$ everywhere for $z \in M$, where $\{\det Jf_j(z)\}_{j=1}^{\infty}$ is normal because of the Cauchy estimates. This means, for any $z \in M$, $f_j(z)$ does not approach $\partial\Omega$. So by Theorem 1.6, we find the limit F of f_j is 1-1. Moreover, this means M is taut. $\square$

Lemma 2.1 and Corollary 2.2 tell us that if there exists one point q such that $f_j(q) \to p \in \Omega$, then for any point $z \in M$, we have $f_j(z)$ approaching an interior point of Ω. Furthermore, the limit of f_j forms a biholomorphism. However, this is not the only case. Indeed, sometimes $f_j(q)$ can approach a boundary point of Ω, and this brings trouble for getting the biholomorphism. For example, the image of $F = \lim_{j\to\infty} f_j$ might be just a constant map into a boundary point under some circumstances. The constant map of course cannot be a biholomorphism. What's behind this phenomenon is that under this situation $\det Jf_j(q) \to 0$ as $j \to \infty$. Thus, we need to compose each f_j with a biholomorphic map ϕ_j so that the resulting map $\det J\phi_j \circ f_j$ has a nonzero limit. To find the appropriate ϕ_j we need to estimate the speed of decay for $\det Jf_j(q)$. It appears the speed of decay can be arbitrary, but indeed, the decay is constrained by the location of $f_j(q)$ due to an application of the Schwarz lemma as follows. The following proposition is one of our main techniques.

Proposition 2.3. *Let M be an m-dimensional Hermitian manifold with a real bisectional curvature bounded from above by a negative number $-K$. Assume M is a monotone union of $\Omega \subset \mathbb{C}^m$ via f_j where Ω is a bounded domain in $\mathbb{C}^m$ with a complete Bergman metric of which the Ricci curvature is bounded from below by a negative number $-k$. We also assume there exists a point $q \in M$ so that $f_j(q) \to p \in \partial\Omega$ where p is strongly pseudoconvex. Then $|Jf_j(q)|/\delta(f_j(q))^{(m+1)/2} \gtrsim \eta$ for some $\eta > 0$, where δ is the Euclidean distance function of Ω, i.e., $\delta(z) = \mathrm{dist}(z, \partial\Omega)$.*

Proof. Applying Theorem 1.6 for f_j^{-1}, we have $(f_j^{-1})^* g_M \leq C g_\Omega$ for some $C > 0$ where g_M is the metric on M and g_Ω is the Bergman metric of Ω. In local coordinates, we have for any tangent vector $X_o \in T_o\Omega$ at $o \in \Omega$

$$((f_j^{-1})_* X_o)' G_M(f_j^{-1}(o))(f_j^{-1})_* X_o \leq C X_o' G_\Omega(o) X_o,$$

where we denote the conjugate transpose by $'$ and matrices of g_M and g_Ω by G_M and G_Ω, respectively. For each $j > 0$, we let $o = f_j(q)$ and have

$$((f_j^{-1})_* X_{f_j(q)})' G_M(f_j^{-1}(f_j(q)))(f_j^{-1})_* X_{f_j(q)} \leq C X'_{f_j(q)} G_\Omega(f_j(q)) X_{f_j(q)}.$$

Without loss of generality, we pick up the coordinates on M at q so that G_Ω is the identity matrix at q. Hence, $(Jf_j^{-1}(f_j(q)))' Jf_j^{-1}(f_j(q)) \le CG_\Omega(f_j(q))$ and by the Minkowski determinant theorem, we also have

$$|\det Jf_j^{-1}(f_j(q))|^2 \le C|\det G_\Omega(f_j(q))|. \tag{4}$$

But G_Ω is a metric around a strongly pseudoconvex point p, so by [Fefferman 1974], it is equivalent to the $\partial\bar\partial(\log\delta)$ up to nonzero constant. Moreover, by computing the second-order Taylor expansion of δ at p, we also have

$$|\det G_\Omega(o)| \le \frac{c_0}{\delta(o)^{m+1}}$$

for some $c_0 > 0$, when o is close to p. Again, putting $o = f_j(q)$, we have

$$|\det G_\Omega(f_j(q))| \le \frac{c_0}{\delta(f_j(q))^{m+1}} \tag{5}$$

for sufficiently big $j > 0$. Since $\det Jf_j^{-1}(f_j(q)) \cdot \det Jf_j(q) = 1$, we have, by (4) and (5), that $|\det Jf_j(q)|/\delta(f_j(q))^{(m+1)/2} > 1/\sqrt{c_0C}$ for sufficient $j > 0$. We let $\eta = 1/\sqrt{c_0C}$, and thus get the desired result. □

Another technique in this section was motivated by a simple observation in one variable.

Lemma 2.4. *Suppose there is a family of Möbius transforms on the unit disc $\psi_j(z) = (z+\alpha_j)/(1+\bar\alpha_j z)$ where $\alpha_j \in \mathbb{R}$ and $\alpha_j \to 1$. Fixing $s \in (0,1)$, we define the disc contained in $\mathbb{D}$:*

$$\mathcal{D}_s := \{z \in \mathbb{C} : |z-b| < 1-b\}$$

where $s = 1-b$. Then $\psi_j^{-1}(\mathcal{D}_s) \to \mathbb{D}$ as $j \to \infty$ in the sense of convergence in increasing subsets.

Proof. We compute the preimage $\psi_j^{-1}(\mathcal{D}_s)$. By calculation, we see that

$$\left|\frac{z+\alpha_j}{1+\bar\alpha_j z} - b\right| < 1-b$$

is equivalent to the inequality

$$\left|z + \frac{(\alpha_j - b)(1-\alpha_j b) - (1-b)^2\alpha_j}{|1-\bar\alpha_j b|^2 - (1-b)^2|\alpha_j|^2}\right|^2 < \frac{|1-b|^2 - |\alpha_j - b|^2}{|1-\bar\alpha_j b|^2 - (1-b)^2|\alpha_j|^2} + \frac{|(\alpha_j - b)(1-\alpha_j b) - (1-b)^2\alpha_j|^2}{(|1-\bar\alpha_j b|^2 - (1-b)^2|\alpha_j|^2)^2}.$$

This is a disc centered at

Lemma 2.5. *Suppose there is a family of automorphisms*

$$\psi_j(z,w)=\left(\frac{z+a_j}{1+\bar{a}_j z},\ \frac{\sqrt{1-|a_j|^2}}{1+\bar{a}_j z}w\right)$$

of the unit ball $\mathbb{B}^2$, where $a_j\in\mathbb{R}$ and $a_j\to 1$. Fixing $s\in(0,1)$, we define a ball contained in $\mathbb{B}^2$:

$$\mathcal{B}_s:=\big\{(z,w)\in\mathbb{C}:|(z,w)-(b,0)|<1-b\big\},$$

where $s=1-b$. Then $\psi_j^{-1}(\mathcal{B}_s)\to\mathbb{B}^m$ as $j\to\infty$ in the sense of convergence in increasing subsets.

Proof. We want to compute the preimage $\psi_j^{-1}(\mathcal{B}_s)$. For this, we need to calculate the $(z,w)\in\mathbb{C}^2$, such that $|\psi_j^{-1}(z,w)-(b,0)|<1-b$. By calculation, this is equivalent to the inequality

$$\left|z+\frac{(a_j-b)(1-a_jb)-(1-b)^2a_j}{|1-\bar{a}_jb|^2-(1-b)^2|a_j|^2}\right|^2+\left|\frac{\sqrt{1-|a_j|^2}}{\sqrt{|1-\bar{a}_jb|^2-(1-b)^2|a_j|^2}}w\right|^2$$
$$<\frac{(1-b)^2-|a_j-b|^2}{|1-\bar{a}_jb|^2-(1-b)^2|a_j|^2}+\frac{|(a_j-b)(1-a_jb)-(1-b)^2a_j|^2}{(|1-\bar{a}_jb|^2-(1-b)^2|a_j|^2)^2}. \tag{6}$$

Again, as in the previous lemma, one can see the formula in (6) approaches

$$|z|^2+|w|^2<1.$$ □

Due to symmetries of balls, one can see the following lemma is also true.

Lemma 2.6. *Suppose there is a family of automorphisms*

$$\psi_j(z,w)=\left(\frac{\sqrt{1-|a_j|^2}}{1+\bar{a}_j w}z,\ \frac{w+a_j}{1+\bar{a}_j w}\right)$$

of the unit ball $\mathbb{B}^2$ where $a_j\in\mathbb{R}$ and $a_j\to -1$. Fixing $s\in(0,1)$, we define a ball contained in $\mathbb{B}^2$:

$$\mathcal{B}_s:=\{(z,w)\in\mathbb{C}:|(z,w)-(0,b)|<1+b\}$$

where $s=1+b$. Then $\psi_j^{-1}(\mathcal{B}_s)\to\mathbb{B}^m$ as $j\to\infty$ in the sense of convergence in increasing subsets.

Proof of Theorem 1. Let $q\in M$ and $f_j(q)\to p$ as $j\to\infty$. There are two possibilities for the location of q: $q\in E_n$ or $q\in\partial E_n$.

If $p\in E_n$, then by Lemma 2.1, M is biholomorphic to E_n. Now we analyze the cases that $p\in\partial E_n$. Suppose that $f_j(q)=(a_j,b_j)$. We define

$$\psi_j(z,w) = \left(\frac{z-a_j}{1-\bar{a}_j z}, e^{-i\theta_j} \frac{\sqrt[2n]{1-|a_j|^2}}{\sqrt[n]{1-\bar{a}_j z}} w \right).$$

Here ψ_j is a family of automorphisms of E_n and θ_j is defined so that $\psi_j \circ f_j(p) = (0, b'_j)$ with $b'_j \in \mathbb{R}$. Since $(0, b'_j) \in E_n$, by the boundedness we have that $(0, b'_j) \to (0, b'_0)$, where $-1 \leq b'_0 \leq 1$. If $b'_0 \in (-1, 1)$, then $(0, b'_0) \in E_n$. And then by Lemma 2.1 for $\psi_j \circ f_j$, we know that M is biholomorphic to E_n. If $b'_0 = 1$ or -1, we discuss it as follows.

Without loss of generality, we now assume $b'_0 = -1$. This means it approaches a strongly pseudoconvex point $p_0 = (0, -1)$. The ellipsoid E_n, by translation, has a defining function

$$\rho(z,w) = |w-1|^2 - \sqrt[n]{1-|z|^2} = -2\,\mathrm{Re}\, w + |w|^2 + \frac{1}{n}|z|^2 + o(|z|^2).$$

Here, the point p_0 has been translated to $(0, 0)$.

On the other hand, we define $\mathcal{B}_s := \{(z,w) \in \mathbb{C}^2 : |z|^2 + |w|^2 - 2\,\mathrm{Re}\, w < 0\}$. It is not hard to see $\mathcal{B}_s$ is a ball centered at $(0, 1)$ with radius 1. We also define $\mathcal{B}_l := \{(z,w) \in \mathbb{C}^2 : (1/n)|z|^2 + (1/n)|w|^2 - 2\,\mathrm{Re}\, w < 0\}$. We can see that $\mathcal{B}_l$ is a ball centered at $(0, n)$ with radius n. So $\mathcal{B}_s \subset E_n \subset \mathcal{B}_l$, and they are tangent to each other at $(0, 0)$. We translate and rescale $\mathcal{B}_l$, $\mathcal{B}_s$, and E_n so that $\mathcal{B}_l = \mathbb{B}^2$. This setup is good for applying Lemma 2.6 to our situation. Due to the translation and rescaling, $\psi_j \circ f_j(p)$ becomes $(0, (b'_j + 1 - n)/n)$ and p_0 is once again relocated at $(0, -1)$. Since p_0 is strongly pseudoconvex, by Proposition 2.3, we see there exists $\eta > 0$ so that

$$|J(\psi_j \circ f_j)(q)| \gtrsim \eta \delta(\psi_j \circ f_j(q))^{3/2}.$$

This implies that

$$|J(\psi_j \circ f_j)(q)| \gtrsim \frac{\eta}{n}(1 - |b'_j|)^{3/2},$$

where the n is due to the rescaling $\mathcal{B}_l$ into $\mathbb{B}^2$. We define a family of automorphisms of $\mathcal{B}_l = \mathbb{B}^2$,

$$\phi_j = \left(\frac{\sqrt{1-(b'_j)^2}}{1+b'_j w} z, \frac{w+b'_j}{1+b'_j w} \right).$$

and consequently, their inverses are

$$\phi_j^{-1} = \left(\frac{\sqrt{1-(b'_j)^2}}{1-b'_j w} z, \frac{w-b'_j}{1-b'_j w} \right).$$

Consider

$$J\phi_j^{-1}(z,w)=\frac{(1-(b'_j)^2)^{3/2}}{(1-b'_jw)^3},\qquad (J\phi_j^{-1})(\psi\circ f_j(q))=\frac{(1-(b'_j)^2)^{3/2}}{(1-b'_j(b'_j+1-n)/n)^3}.$$

We see that

$$\begin{aligned}\det(J(\phi_j^{-1}\circ\psi_j\circ f_j)(q))&=\det((J\phi_j^{-1})(\psi_j\circ f_j(q)))\det(J(\psi_j\circ f_j)(q))\\&\gtrsim\frac{\eta}{n}\frac{(1-(b'_j)^2)^3}{(1-b'_j(b'_j+1-n)/n)^3},\end{aligned}$$

where the last term is bounded below by a positive number. This can be seen by a calculation using l'Hôpital's rule on $x=b'_j\to -1$.

Thus, the limit F of $\phi_j^{-1}\circ\psi_j\circ f_j$ has nontrivial image. Moreover, the image of F is $\mathbb{B}^2$ because by Lemma 2.6, $\phi_j^{-1}(\mathcal{B}_s)\subset\phi_j^{-1}(E_n)$ and $\phi_j^{-1}(\mathcal{B}_s)$ grows up to $\mathcal{B}_l=\mathbb{B}^2$.

Finally, we check the injectivity of F. The readers are reminded that the Bergman metric on E_n is invariant under ϕ_j^{-1}. Suppose there are $z',z''\in M$ so that

$$\lim_{j\to\infty}\phi_j^{-1}\circ\psi_j\circ f_j(z')=\lim_{j\to\infty}\phi_j^{-1}\circ\psi_j\circ f_j(z'').$$

We can find big $N>0$ so that for all $j>N$,

$$\phi_j^{-1}\circ\psi_j\circ f_j(z')\in\phi_N^{-1}(E_n),\qquad \phi_j^{-1}\circ\psi_j\circ f_j(z'')\in\phi_N^{-1}(E_n).$$

Consequently, by Theorem 1.6, we have that

$$d_M(z',z'')\le Cd_{\phi_N^{-1}(E_n)}(\phi_j^{-1}\circ\psi_j\circ f_j(z'),\phi_j^{-1}\circ\psi_j\circ f_j(z'')).$$

The assumption that $\lim_{j\to\infty}\phi_j^{-1}\circ\psi_j\circ f_j(z')=\lim_{j\to\infty}\phi_j^{-1}\circ\psi_j\circ f_j(z'')$ implies $z'=z''$. This proves the injectivity of F. $\square$

Without much effort, one can show the following corollary.

Corollary 2.7. *Let M be an m-dimensional Hermitian manifold with a real bisectional curvature bounded from above by a negative number $-K$, and assume M is a monotone union of balls with the same dimension. Then M is biholomorphic onto $\mathbb{B}^m$.*

In [Liu 2017], the author defined a generalized bidisc $\mathbb{D} \rtimes_\theta \mathbb{H}^+ := \{(z, w) : z \in \mathbb{D}, w \in e^{i\theta(z)}\mathbb{H}^+\}$. It has a noncompact automorphism group and shares some properties with the bidisc. Indeed, when $\theta(z)$ is a zero function, $\mathbb{D} \rtimes_\theta \mathbb{H}^+$ is biholomorphic to a bidisc.

In this section, we prove that the generalized bidisc cannot have a complete Kähler metric with holomorphic bisectional curvature bounded by two negative numbers. This is a result of Yang type. Recall that Yang's theorem [1976] on bidiscs has certain requirements on both variables of the bidisc, but in the proof, we show that it is possible to relax the requirement for one of them. Of course similar results for higher dimensions hold for the same reason. But we will not discuss them here. Our proof is modified from [Seo 2012].

Proof of Theorem 2. We assume the conclusion is not true. Let us denote the Poincaré metric of $\mathbb{D}$ by g and the complete Kähler metric on $\mathbb{D} \rtimes_\theta \mathbb{H}^+$ by h. For each z, we define $i_z(w) = (z, ie^{i\theta(z)}(1+w)/(1-w))$ from $\mathbb{D}$ onto $e^{i\theta(z)}\mathbb{H}^+$. We get $i^*h \le (4/c)g$ by the Schwarz lemma of Yau [1978] because the Ricci curvature of $\mathbb{D}$ is -4. Thus,

$$(7)\quad \begin{pmatrix} 0 & \dfrac{2ie^{i\theta(z)}}{(1-w)^2} \end{pmatrix} \begin{pmatrix} h_{1\bar{1}}\left(z, ie^{i\theta}\dfrac{1+w}{1-w}\right) & h_{1\bar{2}}\left(z, ie^{i\theta}\dfrac{1+w}{1-w}\right) \\ h_{2\bar{1}}\left(z, ie^{i\theta}\dfrac{1+w}{1-w}\right) & h_{2\bar{2}}\left(z, ie^{i\theta}\dfrac{1+w}{1-w}\right) \end{pmatrix} \begin{pmatrix} 0 \\ \dfrac{-2ie^{-i\theta(z)}}{(1-\bar{w})^2} \end{pmatrix}$$
$$= h_{2\bar{2}}\left(z, ie^{i\theta(z)}\frac{1+w}{1-w}\right)\frac{4}{|1-w|^4} \le \frac{4}{c(1-|w|^2)^2}.$$

The last inequality gives

$$h_{2\bar{2}}\left(z, ie^{i\theta(z)}\frac{1+w}{1-w}\right) \le \frac{|1-w|^4}{c(1-|w|^2)^2} \le \frac{16}{c(1-|w|^2)^2}.$$

Since $k < \pi$, there exists $\epsilon > 0$ such that $k + \epsilon < \pi$. Because of $0 \le \theta(z) < k$, the following is true: $(z, e^{i(k+\epsilon/2)}) \in \mathbb{D} \rtimes_\theta \mathbb{H}^+$. We also have, for all $z \in \mathbb{D}$,

$$(8)\qquad \frac{\epsilon}{2} < k + \frac{\epsilon}{2} - \theta(z) < k + \frac{\epsilon}{2} < k + \epsilon < \pi.$$

We fix $w_0 = (e^{i(k+\epsilon/2)-\theta(z)} - i)/(e^{i(k+\epsilon/2-\theta(z))} + i)$ for all $z \in \mathbb{D}$, and by the inequality (8), we can see $|1 - |w_0|| > \eta > 0$ for some positive number η depending on ϵ. Also by the inequality (7), we have

$$h_{2\bar{2}}\left(z, ie^{i\theta(z)}\frac{1+w_0}{1-w_0}\right) = h_{2\bar{2}}(z, e^{i\theta(z)}e^{i((k+\epsilon/2)-\theta(z))}) = h_{2\bar{2}}(z, e^{i(k+\epsilon/2)}) \le \frac{16}{c\eta^2}.$$

Let $F(z) := h_{2\bar{2}}(z, e^{i(k+\epsilon/2)})$. We see F is a real bounded positive function on $\mathbb{D}$. We check its Laplacian with respect to Poincaré metric on $\mathbb{D}$; we have (considering the bound of $R_{2\bar{2}1\bar{1}}$)

$$\begin{aligned}\Delta_g F(z) &= (1-|z|^2)^2 \frac{\partial^2 F}{\partial z\, \partial \bar{z}}(z) \\ &= (1-|z|^2)^2 \left(R_{2\bar{2}1\bar{1}}(z, e^{i(k+\epsilon/2)}) + \sum_{\alpha,\beta=1}^{2} h^{\alpha\bar{\beta}} \frac{\partial h_{2\bar{\beta}}}{\partial z} \frac{\partial h_{\alpha\bar{2}}}{\partial \bar{z}} \right) \\ &\geq c(1-|z|^2)^2 h_{2\bar{2}}(z, e^{i(k+\epsilon/2)}) h_{1\bar{1}}(z, e^{i(k+\epsilon/2)}) \\ &= cF(z)(1-|z|^2)^2 h_{1\bar{1}}(z, e^{i(k+\epsilon/2)}),\end{aligned}$$

because $\sum_{\alpha,\beta=1}^{2} h^{\alpha\bar{\beta}}(\partial h_{2\bar{\beta}}/\partial z)(\partial h_{\alpha\bar{2}}/\partial \bar{z})$ is nonnegative. Let $\pi : \mathbb{D} \rtimes_\theta \mathbb{H}^+ \to \mathbb{D}$, $\pi(z, w) = z$. We also have $\pi^* g \leq (d/4)h$, which is $(1-|z|^2)^2 h_{1\bar{1}}(z, w) \leq 4/d$. Hence, $\Delta_g F(z) \geq (c/d)F$. Calculate

$$\Delta_g \log F(z) = \frac{\Delta_g F(z)}{F(z)} - \frac{|\nabla_g F(z)|^2}{F(z)^2} \geq \frac{2c}{d} - \frac{|\nabla_g F(z)|^2}{F(z)^2}.$$

By Theorem 1.4, a real function T bounded from above on a complete Riemannian manifold M with Ricci curvature bounded below admits a sequence $\{p_k\}_{k=0}^{\infty} \subset M$ such that

$$\lim_{k\to\infty} |\nabla T(p_k)| = 0, \qquad \limsup_{k\to\infty} \Delta T(p_k) \leq 0, \qquad \lim_{k\to\infty} T(p_k) = \sup_M T.$$

Although $\log F(z)$ is a real function bounded from above on $\mathbb{D}$, it can not have such sequence $\{p_k\}_{k=0}^{\infty} \subset \mathbb{D}$. This contradiction completes the proof. ☐

CONCLUSION

A natural question is if we can relax the restriction for $\theta(z)$ in the theorem above.

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