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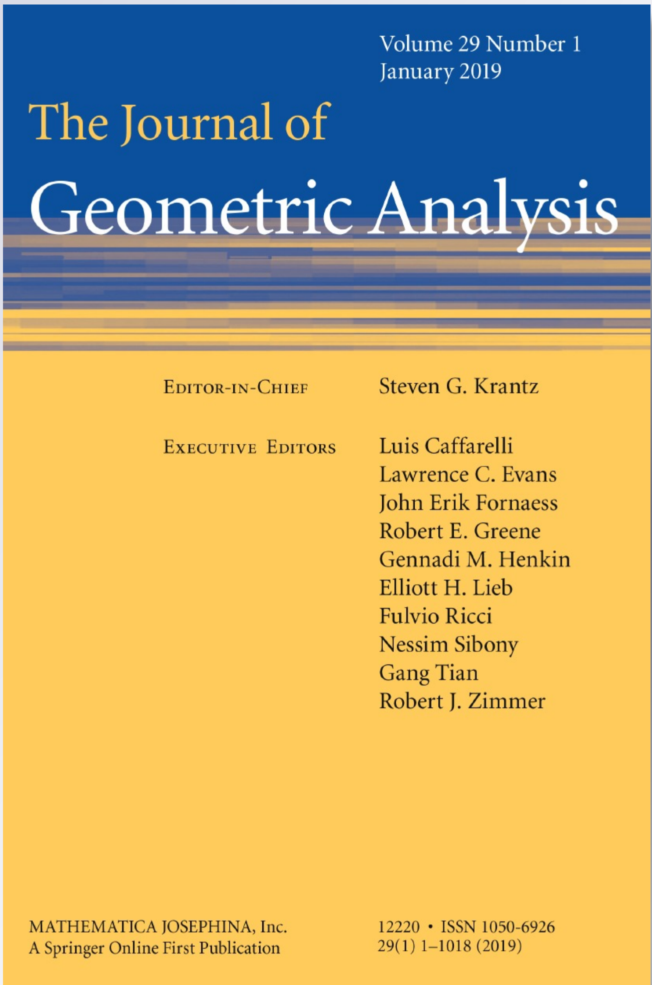

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On the Automorphism Groups of Finite Multitype Models in $\mathbb{C}^n$

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Abstract In this paper, we give an explicit description for the automorphism groups of finite multitype models in $\mathbb{C}^n$.

Keywords Automorphism group · Finite multitype model · Finite type point

Mathematics Subject Classification Primary 32M05 · Secondary 32H02 · 32T25

1 Introduction

For a point $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, we denote by z' the vector of the first $n-1$ components of z. In what follows, we assign weights $\frac{1}{2m_1}, \ldots, \frac{1}{2m_{n-1}}$, 1 to the variables

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$z_1, \ldots, z_{n-1}, z_n$, respectively, and denote by $wt(K) := \sum_{j=1}^{n-1} \frac{k_j}{2m_j}$ the weight of an $(n-1)$-tuple $K = (k_1, \ldots, k_{n-1}) \in \mathbb{Z}_{\geq 0}^{n-1}$.

A real-valued polynomial P on $\mathbb{C}^{n-1}$ is called a *weighted homogeneous polynomial with weight* $(m_1, \ldots, m_{n-1})$ (or simply $(1/m_1, \ldots, 1/m_{n-1})$-*homogeneous*), if

$$P(t^{1/2m_1} z_1, \ldots, t^{1/2m_{n-1}} z_{n-1}) = t P(z_1, \ldots, z_{n-1}) \text{ for all } z' \in \mathbb{C}^{n-1} \text{ and } t > 0.$$

In the case when $m = m_1 = \ldots = m_{n-1}$, then P is called *homogeneous of degree* m. What is more, we note that if $P(z')$ is a $(1/m_1, \ldots, 1/m_{n-1})$-homogeneous polynomial, then

$$P(z') = \sum_{wt(K)+wt(L)=1} a_{KL} z'^K \bar{z}'^L, \tag{1}$$

where $a_{KL} \in \mathbb{C}$ with $a_{KL} = \bar{a}_{LK}$ (see Corollary 1 in Sect. 3).

In this paper, we establish an explicit description for the automorphism group of a finite multitype (in the sense of Catlin) model in $\mathbb{C}^n$ which is defined by

$$M_P = \{z \in \mathbb{C}^n : \text{Re } z_n + P(z') < 0\},$$

where P is a real-valued weighted homogeneous plurisubharmonic polynomial in $\mathbb{C}^{n-1}$ without harmonic terms. The finite multitype hypersurface ∂M_P was defined as a model hypersurface associated to a point of finite Catlin's multitype (see [16]). Moreover, the Lie algebra of all germs of infinitesimal automorphisms of ∂M_P at 0 was explicitly described by Kolar et al. [17] (see also [18]).

The Catlin's multitype has attracted considerable attention, largely due to the invariant property under biholomorphic mappings and the global regularity issue on the $\bar{\partial}$-Neumann problem (cf. [5,6]). For the comparison with other well-known finite type conditions, we refer to [7,25,26] and the references therein. To elaborate our motivation focused on the biholomorphic equivalence problem of the model M_P, we selectively present the following historical background: As a local version of a result by Bedford and Pinchuk [2], it is a well-known result of Gaussier [10] that if a domain $\Omega \subset \mathbb{C}^n$ is convex of D'Angelo finite type near a boundary orbit accumulation point, then Ω is biholomorphically equivalent to a rigid polynomial domain (see [10, Theorem 1]). Recently, a characterization of finite multitype models was also established by Rong and Zhang in [21]. For the case when Ω is strongly pseudoconvex, Wong [24] and Rosay [22] showed that every bounded strongly pseudoconvex domain in $\mathbb{C}^n$ with non-compact automorphism group is biholomorphically equivalent to the complex unit ball. In addition, when $n = 2$, the associated automorphism group of the model M_P was completely determined in [19].

We also consider two special classes of domains D_P and Q_P in $\mathbb{C}^n$ $(n \geq 2)$ defined, respectively, by

$$D_P := \{(z', z_n) \in \mathbb{C}^n : |z_n|^2 + P(z') < 1\};$$
$$Q_P := \{(z', z_n) \in \mathbb{C}^n : \text{Re } z_n + P(z') < 0\},$$

where

$$P(z') = \sum_{wt(K)=wt(L)=1/2} a_{KL} z'^K \bar{z}'^L, \tag{2}$$

where $a_{KL} \in \mathbb{C}$ with $a_{KL} = \bar{a}_{LK}$.

We note that D_P is *bounded* if and only if $P(z') > 0$ for all $z' \in \mathbb{C}^{n-1} \setminus \{0\}$ (cf. [13] and Lemma 6). Moreover, if D_P is bounded, then the automorphism group $\mathrm{Aut}(D_P)$ is non-compact since it contains $\{\phi_{a,\theta}: a \in \Delta, \theta \in \mathbb{R}\}$, where $\phi_{a,\theta}$ is defined by

$$(z', z_n) \mapsto \left(\frac{(1-|a|^2)^{1/2m_1}}{(1-\bar{a}z_n)^{1/m_1}} z_1, \ldots, \frac{(1-|a|^2)^{1/2m_{n-1}}}{(1-\bar{a}z_n)^{1/m_{n-1}}} z_{n-1}, e^{i\theta} \frac{z_n - a}{1-\bar{a}z_n} \right),$$

where $a \in \Delta := \{z \in \mathbb{C}: |z| < 1\}$ and $\theta \in \mathbb{R}$ (see Lemma 7 in Sect. 4). As we can see in the proof of Lemma 7, the weighted homogeneity of P and the special form (2) allow us to obtain that $\phi_{a,\theta} \in \mathrm{Aut}(D_P)$.

Our first aim is to prove that $\mathrm{Aut}(D_P)$ is exactly generated by the set of all above automorphisms and G_P, where G_P is the set of all automorphisms of the form $(z', z_n) \mapsto (Az', z_n)$, where $A = \mathrm{diag}(A_1, \ldots, A_k)$ is a block diagonal matrix with the condition that each A_j $(1 \le j \le k)$ is an invertible $(i_j - i_{j-1}) \times (i_j - i_{j-1})$ matrix and $P(Az') \equiv P(z')$, for integers $i_1, \ldots, i_k$ such that

$$m_1 = \ldots = m_{i_1} > \ldots > m_{i_{j-1}+1} = \ldots = m_{i_j} > \ldots > m_{i_{k-1}+1} = \ldots = m_{i_k} = m_{n-1}.$$

Our first main result is the following theorem.

Theorem 1 *Let P be a real-valued weighted homogeneous plurisubharmonic polynomial in $\mathbb{C}^{n-1}$ given by (2) with a further assumption that $P(z') > 0$ for all $z' \in \mathbb{C}^{n-1} \setminus \{0\}$. Then,* $\mathrm{Aut}(D_P)$ *is generated by G_P and* $\{\phi_{a,\theta}: a \in \Delta, \theta \in \mathbb{R}\}$.

We sketch briefly the main ideas for the proof of Theorem 1 as follows. The positive assumption on P on the set $\mathbb{C}^{n-1} \setminus \{0\}$ implies that D_P is bounded and $|z_n| < 1$ on D_P; hence, the n-th component of $\phi_{a,\theta}$ is contained in the unit disc in $\mathbb{C}$. In addition, we note that any automorphism of D_P can be smoothly extended to the boundary (see [3]). Then the above two facts imply that for any $f \in \mathrm{Aut}(D_P)$, the restriction mapping $f|_{D_P \cap \{z'=0\}} \in \mathrm{Aut}(\Delta)$, where Δ is the unit disc in $\mathbb{C}$. More precisely, the n-th component of f is of the following form:

$$f_n(z) = f_n(0', z_n) = e^{i\theta_n} \frac{z_n - a}{1 - \bar{a}z_n},$$

where $a \in \Delta$ and $\theta_n \in \mathbb{R}$. Replacing f by $\phi_{-a,-\theta_n} \circ f$, we may assume that $f(0) = 0$. Then Lemma 8 in Sect. 4 induced from the weighted homogeneity of the polynomial P, implies that $f \in \mathrm{Aut}(D_P)$ with $f(0) = 0$ must be linear, that is, $f \in G_P$; thus this concludes the proof of Theorem 1.

Furthermore, we note that D_P is biholomorphically equivalent to Q_P, where

$$Q_P := \{(z', z_n) \in \mathbb{C}^n: \mathrm{Re}\, z_n + P(z') < 0\},$$

provided that P is a weighted homogeneous polynomial with weight $(m_1, \dots, m_{n-1})$ given by (2) such that $P(z') > 0$ for all $z' \in \mathbb{C}^{n-1} \setminus \{0\}$ (cf. Theorem 5 in Sect. 4). Consequently, the group $\mathrm{Aut}(Q_P)$ is exactly generated by the translations T_t given by $T_t(z) = (z', z_n + it)$ for $t \in \mathbb{R}$, G_P, and the set of all biholomorphisms of the following form:

$$(z', z_n) \mapsto \left(\frac{(\alpha)^{1/2m_1}}{(1 + i\beta z_n)^{1/m_1}} z_1, \dots, \frac{(\alpha)^{1/2m_{n-1}}}{(1 + i\beta z_n)^{1/m_{n-1}}} z_{n-1}, \frac{\alpha z_n}{1 + i\beta z_n} \right),$$

where $\alpha > 0,\ \beta \in \mathbb{R}$.

Next we discuss our second main result concerning a description for the automorphism group of a finite multitype model. First of all, we recall the definition of WB-domain introduced by Ahn et al. (cf. [1]). A domain M_P in $\mathbb{C}^n$ is called a WB-domain (meaning "*weighted-bumped*") if

$$M_P = \{z \in \mathbb{C}^n : \mathrm{Re}\, z_n + P(z') < 0\},$$

where

(i) P is a real-valued, weighted homogeneous polynomial on $\mathbb{C}^{n-1}$ with weight $(m_1, \dots, m_{n-1})$;
(ii) M_P is strongly pseudoconvex at every boundary point outside the set $\{\partial M_P \cap (\{0'\} \times i\mathbb{R})\}$.

It was also established in [1, Corollary 4.3] that every boundary point of WB-domain M_P admits a peak function for $\mathcal{O}(M_P)$, where $\mathcal{O}(M_P) := \{f : M_P \to \mathbb{C} : f \text{ is holomorphic}\}$. Consequently, its Kobayashi and Bergman metrics are moreover complete (see [1,11]). In addition, there also exists a peak function at infinity for $\mathcal{O}(M_P)$ (cf. Remark 1 in Sect. 2). We especially pay attention to the so-called *generic* model which is not biholomorphically equivalent to any *rotational* model or to any *tubular* model (cf. Definitions 2 and 3 in Sect. 5). Let S_λ $(\lambda > 0)$ and $T_s (s \in \mathbb{R})$ be automorphisms of M_P defined, respectively, by

$$S_\lambda(z) = (\lambda^{1/2m_1} z_1, \dots, \lambda^{1/2m_{n-1}} z_{n-1}, \lambda z_n);\ \ T_s(z) = (z', z_n + is).$$

With the above notations, our second main result is the following theorem.

Theorem 2 *Let M_P be a generic model satisfying that M_P is not biholomorphically equivalent to Q_P and $P(z') > 0$ for all $z' \in \mathbb{C}^{n-1} \setminus \{0\}$. If M_P is a* WB*-domain, then* $\mathrm{Aut}(M_P)$ *is generated by*

$$\{T_t, S_\lambda : t \in \mathbb{R}, \lambda > 0\} \cup G_P.$$

The condition that P is positive on $\mathbb{C}^{n-1} \setminus \{0\}$ plays a substantial role in proving Theorem 2, namely, it is an essential condition of Theorem 6 as a crucial technical lemma for the proof of Theorem 2. Thanks to this condition, the n-th component of any automorphism of M_P can be written as a Möbius transformation. Combining this fact with the invariance of $\overline{M_P}$ under any dilation S_λ with $\lambda > 0$ and comparison of the weighted orders of terms, an explicit form of $\mathrm{Aut}(M_P)$ can be described completely.

The organization of this paper is as follows: In Sect. 2 we recall the concept of the Catlin's multitype and the existence of a peak function at infinity for $\mathcal{O}(M_P)$ is also given. In Sect. 3, we give some basics on weighted homogeneous polynomials. Then, explicit descriptions for $\mathrm{Aut}(D_P)$ and $\mathrm{Aut}(Q_P)$ are given in Sect. 4. Finally, we shall prove Theorem 2 in detail; several examples are also investigated in Sect. 5.

2 Preliminaries

2.1 Catlin's Multitype

For the convenience of the exposition, let us recall *Catlin's multitype* (for more details, we refer to [6,25] and the references therein). Let Ω be a domain in $\mathbb{C}^n$ and ρ be a defining function for Ω near $z_0 \in \partial\Omega$. Let us denote by Λ^n the set of all n-tuples of numbers $\mu = (\mu_1, \ldots, \mu_n)$ such that

(i) $1 \le \mu_1 \le \ldots \le \mu_n \le +\infty$;
(ii) For each j, either $\mu_j = +\infty$ or there is a set of non-negative integers $k_1, \ldots, k_j$ with $k_j > 0$ such that

$$\sum_{s=1}^{j} \frac{k_s}{\mu_s} = 1.$$

A weight $\mu \in \Lambda^n$ is called *distinguished* if there exist holomorphic coordinates $(z_1, \ldots, z_n)$ about z_0 with z_0 maps to the origin such that

$$D^\alpha \overline{D}^\beta \rho(z_0) = 0 \text{ whenever } \sum_{i=1}^{n} \frac{\alpha_i + \beta_i}{\mu_i} < 1.$$

Here D^α and $\overline{D}^\beta$ denote the partial differential operators

$$\frac{\partial^{|\alpha|}}{\partial z_1^{\alpha_1} \ldots \partial z_n^{\alpha_n}} \text{ and } \frac{\partial^{|\beta|}}{\partial \bar{z}_1^{\beta_1} \ldots \partial \bar{z}_n^{\beta_n}},$$

respectively.

Definition 1 The *multitype* $\mathcal{M}(z_0)$ is defined to be the smallest weight $\mathcal{M} = (m_1, \ldots, m_n)$ in Λ^n (smallest in the lexicographic sense) such that $\mathcal{M} \ge \mu$ for every distinguished weight μ.

2.2 Peak Function at Infinity for $\mathcal{O}(M_P)$

Recently, G. Herbort proved the following result.

Theorem 3 (Lemma 3.3 in [12]) *On a* WB-*domain* M_P *there exist a zero-free holomorphic function* F_∞ *and constants* $L_* > 0$ *and* $N \in \mathbb{N}$ *such that*

(i) $-\pi/8 \le \arg \sqrt[N]{F_\infty} \le \pi/8$;
(ii) $L_*^{-1}\hat{\sigma}(z) \le |F_\infty(z)| \le L_*\hat{\sigma}(z)$;
(iii) $1/2\left(L_*^{-1}\hat{\sigma}(z)\right)^{1/N} \le 1/2|F_\infty(z)|^{1/N} \le \mathrm{Re}\ \sqrt[N]{F_\infty(z)} \le \left(L_*\hat{\sigma}(z)\right)^{1/N}$,

where $\hat{\sigma}(z) := \sum_{j=1}^{n-1} |z_j|^{2m_j} + |z_n|$ *for every* $z \in \mathbb{C}^n$.

Remark 1 The function $\varphi(z) := \exp\left(-\frac{1}{\sqrt[N]{F_\infty(z)}}\right)$ is a peak function at infinity for $\mathcal{O}(M_P)$ in the sense that $\varphi \in \mathcal{O}(M_P)$, $|\varphi(z)| < 1$ for every $z \in M_P$ and $\lim_{M_P \ni z \to \infty} \varphi(z) = 1$.

3 Polynomial of Weighted Homogeneous

In this section, we introduce some basic properties of weighted homogeneous polynomials. First of all, Fu et al. [8] proved the following lemma.

Lemma 1 ([8]) *Let* $f(x_1, \ldots, x_r)$ *be a* $\mathcal{C}^\infty$*-function in a neighborhood of the origin in* $\mathbb{R}^r$*. Suppose that there exist* $k_j \in \mathbb{N},\ j = 1, \ldots, r$*, such that*

$$f(t^{1/k_1}x_1, \ldots, t^{1/k_r}x_r) = tf(x_1, \ldots, x_r),$$

for $1 \le t \le 1 + \epsilon$*. Then* f *has the form of the following*

$$f(x_1, \ldots, x_r) = \sum_{l_1,\ldots,l_r} b_{l_1,\ldots,l_r} x_1^{l_1} \ldots x_r^{l_r},$$

where $b_{l_1,\ldots,l_r} \in \mathbb{R}$*, and the sum is taken over all* r*-tuples* $(l_1, \ldots, l_r),\ l_j \in \mathbb{Z},\ l_j \ge 0$*, such that* $\sum_{j=1}^r \frac{l_j}{k_j} = 1$.

From Lemma 1, one can easily establish the following.

Corollary 1 *If* P *is a weighted homogeneous polynomial with weight* $(m_1, \ldots, m_{n-1})$*, then*

$$P(z') = \sum_{wt(K)+wt(L)=1} a_{KL} z'^K \bar{z}'^L,\ \forall z' \in \mathbb{C}^{n-1},$$

where $a_{KL} \in \mathbb{C}$ *with* $a_{KL} = \bar{a}_{LK}$.

Now we prepare one more lemma which is known as *Euler's identity* for weighted homogeneous polynomials as follows.

Lemma 2 *If P is a weighted homogeneous polynomial with weight $(m_1, \ldots, m_{n-1})$, then*

$$2\mathrm{Re}\sum_{j=1}^{n-1} \frac{\partial P}{\partial z_j}\frac{z_j}{2m_j} = P(z'),\ \forall z' \in \mathbb{C}^{n-1}. \tag{3}$$

The proof of this lemma easily follows from the weighted homogeneity condition of P, and we omit it.

Notice that any WB-domain is of D'Angelo finite type. Consequently, its boundary is variety-free at any boundary point, and hence the set $\{P = 0\}$ contains no non-trivial analytic set passing through the origin. The following lemma assures the uniqueness of Euler's identity for non-degenerate weighted homogeneous polynomials.

Lemma 3 *Let P be a weighted homogeneous polynomial with weight $(m_1, \ldots, m_{n-1})$ given by (1) such that $\{P = 0\}$ contains no non-trivial analytic set passing through the origin. If there exist $\alpha_1, \ldots, \alpha_{n-1} \in \mathbb{R}$ such that*

$$2\mathrm{Re}\sum_{j=1}^{n-1} \alpha_j \frac{\partial P}{\partial z_j}\frac{z_j}{2m_j} = P(z'),\ \forall z' \in \mathbb{C}^{n-1}, \tag{4}$$

then $\alpha_1 = \ldots = \alpha_{n-1} = 1$.

Proof Suppose that there exist $\alpha_1, \ldots, \alpha_{n-1} \in \mathbb{R}$ such that (4) holds. Then, from Lemma 2 we immediately have

$$2\mathrm{Re}\sum_{j=1}^{n-1} \frac{\partial P}{\partial z_j}\frac{z_j}{2m_j} = P(z'),\ \forall z' \in \mathbb{C}^{n-1}.$$

Hence, combining this fact with (4) one gets

$$2\mathrm{Re}\sum_{j=1}^{n-1} (1-\alpha_j) \frac{\partial P}{\partial z_j}\frac{z_j}{2m_j} = 0,\ \forall z' \in \mathbb{C}^{n-1};$$

this relation yields $\alpha_1 = \ldots = \alpha_{n-1} = 1$ since the set $\{P = 0\}$ contains no non-trivial analytic set passing through the origin, as desired. □

Note that Kim and the first author in [14, Lemma 4] proved that $\mathrm{Re}\,(izR(z)) = 0$ if and only if $R(z) = R(|z|)$ provided that $R \in \mathcal{C}^1(\Delta_\epsilon)$ for some $\epsilon > 0$, where $\Delta_\epsilon := \{z \in \mathbb{C} \colon |z| < \epsilon\}$. The following lemma generalizes this result to the case of weighted homogeneous polynomials.

Lemma 4 *Let P be a weighted homogeneous polynomial with weight $(m_1, \ldots, m_{n-1})$. Then*

$$2\mathrm{Re}\sum_{j=1}^{n-1} i \frac{\partial P}{\partial z_j}\frac{z_j}{2m_j} = 0,\ \forall z' \in \mathbb{C}^{n-1},$$

if and only if

$$P(z') = \sum_{wt(K)=wt(L)=1/2} a_{KL} z'^K \bar{z'}^L, \ \forall z' \in \mathbb{C}^{n-1}, \tag{5}$$

where $a_{KL} \in \mathbb{C}$ *with* $a_{KL} = \bar{a}_{LK}$.

Proof For the proof of "only if" part, it suffices to prove the assertion for $P(z') = az'^I \bar{z'}^J + \bar{a} z'^J \bar{z'}^I$ ($a \in \mathbb{C}^*$), where I, J are $(n-1)$-tuples with $wt(I) + wt(J) = 1$. Indeed, since

$$2\text{Re} \sum_{k=1}^{n-1} i \frac{\partial P}{\partial z_k} \frac{z_k}{2m_k} = 0, \ \forall z' \in \mathbb{C}^{n-1},$$

we have

$$\begin{aligned} 0 &= \left(\sum_{k=1}^{n-1} \frac{i_k}{2m_k} \right) 2\text{Re} \left(iaz'^I \bar{z'}^J \right) + \left(\sum_{k=1}^{n-1} \frac{j_k}{2m_k} \right) 2\text{Re} \left(i\bar{a} z'^J \bar{z'}^I \right) \\ &= \left(\sum_{k=1}^{n-1} \frac{i_k}{2m_k} - \sum_{k=1}^{n-1} \frac{j_k}{2m_k} \right) 2\text{Re} \left(iaz'^I \bar{z'}^J \right) \end{aligned}$$

for all $z' \in \mathbb{C}^{n-1}$. This yields

$$wt(I) = \sum_{k=1}^{n-1} \frac{i_k}{2m_k} = \sum_{k=1}^{n-1} \frac{j_k}{2m_k} = wt(J)$$

or

$$\text{Re} \left(iaz'^I \bar{z'}^J \right) = 0, \ \forall z' \in \mathbb{C}^{n-1}.$$

The first case indicates that the conclusion holds. Then the conclusion follows immediately since we must have $I = J$ in the latter case.

The proof of "if" part directly follows from differentiating both sides of (5) with respect to z_j, $1 \leq j \leq n-1$. This completes the proof of this lemma. □

In the remaining of this section, we shall recall some known results on the holomorphic extension of a biholomorphism to a neighborhood of a given boundary point, and then we prove a key lemma which will be used in proving Theorem 2. First of all, we define the cluster set as follows. If $f : D \to \mathbb{C}^N$ is a holomorphic function on a domain $D \subset \mathbb{C}^n$ and $z_0 \in \partial D$, we denote by $\mathcal{C}(f, z_0)$ the *cluster set* of f at z_0:

$$\mathcal{C}(f, z_0) = \{w \in \mathbb{C}^N : w = \lim f(z_j), z_j \in D, \text{and } \lim z_j = z_0\}.$$

A. B. Sukhov [23] proved the following:

Lemma 5 (See Corollary 1.4 in [23]) *Suppose that D and G are $\mathcal{C}^\infty$-smooth domains in $\mathbb{C}^n$. Suppose that D and G are pseudoconvex of finite type near $z_0 \in \partial D$ and $w_0 \in \partial G$, respectively. Let f be a biholomorphic mapping from D onto G such that $w_0 \in \mathcal{C}(f, z_0)$. Then f and f^{-1} extend smoothly to ∂D in some neighborhoods of the points z_0 and w_0, respectively.*

Concerning proper holomorphic maps between bounded domains, we recall the following theorem given in [20, Theorem 2']

Theorem 4 *Let $D, D' \subset \mathbb{C}^n, n \geq 2$, be bounded domains and let $f : D \to D'$ be a proper holomorphic map such that f extends as a holomorphic correspondence to a neighborhood $U \subset \mathbb{C}^n$ of a point $a \in \partial D$. Suppose that $\partial D \cap U$, $\partial D' \cap U'$ are real analytic hypersurfaces of finite type, where $U' \subset \mathbb{C}^n$ is a neighborhood of $f(a) \in \partial D'$. Then f extends holomorphically to a (possibly smaller) neighborhood of $a \in \partial D$.*

As a generalization of a result in [4, Lemma 3.2] considered in $\mathbb{C}^2$, we have the following proposition in $\mathbb{C}^n$ which is a main ingredient in proving Theorem 2.

Proposition 1 *Let M_P and M_Q be two* WB*-domains. Suppose that $\psi : M_P \to M_Q$ is a biholomorphism. Then there exist $t_0 \in \mathbb{R}$ and $z_0 \in \partial M_Q$ such that ψ and ψ^{-1} extend to be holomorphic in neighborhoods of $(0, it_0)$ and z_0, respectively.*

Proof Thanks to Remark 1, there exists a holomorphic function ϕ on M_Q which is continuous on $\overline{M_Q}$ such that $|\phi| < 1$ for $z \in M_Q$ and tends to 1 at infinity. Let $\psi : M_P \to M_Q$ be a biholomorphism. We claim that there exists $t_0 \in \mathbb{R}$ such that $\liminf_{x \to 0^-} |\psi(0', x + it_0)| < +\infty$. Indeed, if this would not be the case, the function $\phi \circ \psi$ would be equal to 1 on the half plane $\{(z', z_n) \in \mathbb{C}^n : \text{Re } z_n < 0, z' = 0\}$ and this is impossible since $|\phi| < 1$ for $|z| \gg 1$. Therefore, we may assume that there exists a sequence $\{x_k\}$ such that $x_k < 0$, $\lim_{k\to\infty} x_k = 0$ and $\lim_{k\to\infty} \psi(0', x_k + it_0) = z_0 \in \partial M_Q$. Hence, the conclusion follows from Lemma 5 and Theorem 4. □

4 Automorphism Groups of D_P and Q_P

This section is devoted to the explicit descriptions for the automorphism groups of D_P and Q_P, where D_P and Q_P are, respectively, defined by

$$D_P := \{(z', z_n) \in \mathbb{C}^n : |z_n|^2 + P(z') < 1\};$$
$$Q_P := \{(z', z_n) \in \mathbb{C}^n : \text{Re } z_n + P(z') < 0\},$$

where

$$P(z') = \sum_{wt(K)=wt(L)=1/2} a_{KL} z'^K \bar{z}'^L, \tag{6}$$

where $a_{KL} \in \mathbb{C}$ with $a_{KL} = \bar{a}_{LK}$.

It is well-known that if D_P is bounded, then $P(z') \geq 0$ for all $z' \in \mathbb{C}^{n-1}$ (cf. [13]). Moreover, we have the following lemma, which is a generalization of this fact.

Lemma 6 *Let P be a weighted homogeneous polynomial with weight $(m_1, \ldots, m_{n-1})$ given by (1). Then, the domain $\widetilde{D}_P$, defined by*

$$\widetilde{D}_P := \{(z', z_n) \in \mathbb{C}^n : |z_n|^2 + P(z') < 1\},$$

is bounded in $\mathbb{C}^n$ if and only if $P(z') > 0$ for all $z' \in \mathbb{C}^{n-1} \setminus \{0\}$.

Proof Let P be a weighted homogeneous polynomial with weight $(m_1, \ldots, m_{n-1})$ given by (1). First of all, we shall prove the "only if" part of the lemma. Suppose that $\widetilde{D}_P$ is bounded. Then, one can show that $P(z') > 0$ for all $z' \in \mathbb{C}^{n-1} \setminus \{0\}$: suppose otherwise. Then, there exists a point $z' \in \mathbb{C}^{n-1} \setminus \{0\}$ such that $P(z') \leq 0$. Since P is a weighted homogeneous polynomial with weight $(m_1, \ldots, m_{n-1})$, it follows that

$$P(t^{1/2m_1} z_1, \ldots, t^{1/2m_{n-1}} z_{n-1}) = t P(z_1, \ldots, z_{n-1}) \leq 0, \ \forall t > 0.$$

Then, this yields $(t^{1/2m_1} z_1, \ldots, t^{1/2m_{n-1}} z_{n-1}, 0) \in \widetilde{D}_P$ for all $t > 0$, which contradicts the boundedness of $\widetilde{D}_P$. Hence, we obtain $P(z') > 0$ for all $z' \in \mathbb{C}^{n-1} \setminus \{0\}$.

Next, we shall prove the "if" part of the lemma. Suppose that $P(z') > 0$ for all $z' \in \mathbb{C}^{n-1} \setminus \{0\}$. Then, since $\widetilde{D}_P \subset \{(z', z_n) \in \mathbb{C}^n : 0 \leq P(z') < 1;\ |z_n| < 1\}$, it suffices to show that the domain $\{z' \in \mathbb{C}^{n-1} : 0 < P(z') < 1\}$ is bounded. Aiming for a contradiction, suppose that there exists a sequence $\{z'^k\}_{k=1}^\infty \subset \{z' \in \mathbb{C}^{n-1} : 0 < P(z') < 1\}$ such that $z'^k := (z_1^k, \ldots, z_{n-1}^k) \to \infty$ as $k \to \infty$. Choose a sequence $\{t_k\}_{k=1}^\infty$ such that each t_k is positive and $\|(t_k^{1/2m_1} z_1^k, \ldots, t_k^{1/2m_{n-1}} z_{n-1}^k)\| = 1$, where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{C}^{n-1}$. Notice that $t_k \to 0^+$ as $k \to \infty$. Then, since P is a weighted homogeneous polynomial with weight $(m_1, \ldots, m_{n-1})$, it follows that

$$P(t_k^{1/2m_1} z_1^k, \ldots, t_k^{1/2m_{n-1}} z_{n-1}^k) = t_k P(z_1^k, \ldots, z_{n-1}^k) \to 0$$

as $k \to \infty$, which is absurd since P is continuous on the sphere $\{z' \in \mathbb{C}^{n-1} : \|z'\| = 1\}$.

Altogether, the proof of this lemma is complete. □

We note that $\mathrm{Aut}(D_P)$ is non-compact by virtue of the following lemma.

Lemma 7 *Let P be a weighted homogeneous polynomial with weight $(m_1, \ldots, m_{n-1})$ given by (6) such that $P(z') > 0$ for all $z' \in \mathbb{C}^{n-1} \setminus \{0\}$. Then,* $\mathrm{Aut}(D_P)$ *contains the following automorphisms $\phi_{a,\theta}$ defined by*

$$(z', z_n) \mapsto \left(\frac{(1 - |a|^2)^{1/2m_1}}{(1 - \bar{a} z_n)^{1/m_1}} z_1, \ldots, \frac{(1 - |a|^2)^{1/2m_{n-1}}}{(1 - \bar{a} z_n)^{1/m_{n-1}}} z_{n-1}, e^{i\theta} \frac{z_n - a}{1 - \bar{a} z_n} \right),$$

where $a \in \Delta$ and $\theta \in \mathbb{R}$.

Proof Indeed, a direct computation shows that

$$\left|\frac{z_n - a}{1 - \bar{a} z_n}\right|^2 - 1 = \frac{|z_n - a|^2 - |1 - \bar{a} z_n|^2}{|1 - \bar{a} z_n|^2}$$
$$= \frac{|z_n|^2 + |a|^2 - 1 - |a|^2 |z_n|^2}{|1 - \bar{a} z_n|^2}$$
$$= \frac{(|z_n|^2 - 1)(1 - |a|^2)}{|1 - \bar{a} z_n|^2}.$$

Moreover, since P has the form as in (6), it follows that

$$P(\tilde{\phi}_{a,\theta}(z)) = \frac{1 - |a|^2}{|1 - \bar{a} z_n|^2} P(z'),$$

where $\phi_{a,\theta}(z) = (\tilde{\phi}_{a,\theta}(z), (\phi_{a,\theta})_n(z))$. Therefore, one can deduce that

$$|(\phi_{a,\theta})_n(z)|^2 - 1 + P(\tilde{\phi}_{a,\theta}(z)) < 0$$

if and only if

$$|z_n|^2 - 1 + P(z') < 0.$$

Hence, the conclusion can be derived easily from the previous relation. □

In what follows, let $i_1, \ldots, i_k$ be integers such that $m_1 = \ldots = m_{i_1} > \ldots > m_{i_{j-1}+1} = \ldots = m_{i_j} > \ldots > m_{i_{k-1}+1} = \ldots = m_{i_k} = m_{n-1}$. Denote by G_P the set of all automorphisms of the form (Az', z_n), where $A = \text{diag}(A_1, \ldots, A_k)$ is an invertible block diagonal matrix such that each A_j $(1 \le j \le k)$ is an $(i_j - i_{j-1}) \times (i_j - i_{j-1})$ matrix and $P(Az') \equiv P(z')$. In addition, denote by $h_s(z)$ a germ at the origin of holomorphic functions with weighted order greater than s $(s > 0)$.

Before proceeding further, we now prepare a crucial technical lemma for the proofs of Theorems 1 and 2.

Lemma 8 *Let P be a weighted homogeneous polynomial with weight $(m_1, \ldots, m_{n-1})$ given by (1) such that $\{P = 0\}$ contains no non-trivial analytic set passing through the origin. Let $\tilde{f} = (f_1, \ldots, f_{n-1})$ be a biholomorphism on a neighborhood of $0 \in \mathbb{C}^{n-1}$ with $\tilde{f}(0) = 0$. If $P(f_1(z'), \ldots, f_{n-1}(z')) = P(z')$ for all z' in a neighborhood of $0 \in \mathbb{C}^{n-1}$, then $\tilde{f}$ can be extended to a linear mapping on $\mathbb{C}^{n-1}$, and moreover the mapping $f(z', z_n) := (\tilde{f}(z'), z_n)$ belongs to G_P.*

Proof Let P be a weighted homogeneous polynomial as above such that

$$P(f_1(z'), \ldots, f_{n-1}(z')) = P(z')$$

for all z' in a neighborhood of $0 \in \mathbb{C}^{n-1}$. Let us denote by $U(0)$ such a neighborhood of $0 \in \mathbb{C}^{n-1}$. Without loss of generality, we may assume that

$$m_1 \ge m_2 \ge \ldots \ge m_{n-1}.$$

Moreover, since P is a weighted homogeneous polynomial with weight $(m_1, \dots, m_{n-1})$, it follows that

$$P\left(f_1\left(t^{\frac{1}{2m_1}}z_1, \dots, t^{\frac{1}{2m_{n-1}}}z_{n-1}\right), \dots, f_{n-1}\left(t^{\frac{1}{2m_1}}z_1, \dots, t^{\frac{1}{2m_{n-1}}}z_{n-1}\right)\right) = tP(z') \tag{7}$$

for all $t \in (0, 1)$ and $z' \in U(0)$.

Now we shall prove that $df = \mathrm{Id}$ at the origin. Let us consider the two following cases:

Case 1 $m_1 > m_2 > \dots > m_{n-1}$. Fix a point $z' \in U(0)$. Then, since $t^{\frac{1}{2m_1}} > t^{\frac{1}{2m_2}} > \dots > t^{\frac{1}{2m_{n-1}}}$ for any $t \in (0, 1)$, one has for each $1 \le j \le n-1$

$$f_j(z') = a_{j,j}z_j + h_{1/2m_j}(z'),$$

where $a_{1,1}, \dots, a_{n-1,n-1} \neq 0$. Recall that $h_s(z)$ denotes a germ at the origin of holomorphic functions with weighted order greater than s. Then Eq. (7) becomes

$$P\left(t^{\frac{1}{2m_1}}a_{1,1}z_1 + o\left(t^{\frac{1}{2m_1}}\right), \dots, t^{\frac{1}{2m_{n-1}}}a_{n-1,n-1}z_{n-1} + o\left(t^{\frac{1}{2m_{n-1}}}\right)\right) = tP(z') \tag{8}$$

for all $t \in (0, 1)$ and $z' \in U(0)$. Dividing both sides of (8) by t, it follows that

$$P\left(a_{1,1}z_1 + o\left(t^{\frac{1}{2m_1}}\right)\Big/t^{\frac{1}{2m_1}}, \dots, a_{n-1,n-1}z_{n-1} + o\left(t^{\frac{1}{2m_{n-1}}}\right)\Big/t^{\frac{1}{2m_{n-1}}}\right) = P(z') \tag{9}$$

for all $t \in (0, 1)$ and $z' \in U(0)$. Now, by evaluating the limit as $t \to 0^+$ of the left-hand side of (9), we arrive at

$$P(a_{1,1}z_1, a_{2,2}z_2, \dots, a_{n-1,n-1}z_{n-1}) = P(z') \tag{10}$$

for all $z' \in U(0)$. A similar argument for f^{-1} gives

$$P\left(a_{1,1}^{-1}z_1, a_{2,2}^{-1}z_2, \dots, a_{n-1,n-1}^{-1}z_{n-1}\right) = P(z') \tag{11}$$

for all $z' \in U(0)$. For a fixed point $z' \in \mathbb{C}^{n-1}$, choose a $t > 0$ sufficiently small so that

$$\left(t^{\frac{1}{2m_1}}z_1, \dots, t^{\frac{1}{2m_{n-1}}}z_{n-1}\right) \in U(0).$$

Therefore, by (10) and (11), we have

$$\begin{aligned} P\left(t^{\frac{1}{2m_1}}z_1, \dots, t^{\frac{1}{2m_{n-1}}}z_{n-1}\right) &= P\left(t^{\frac{1}{2m_1}}a_{1,1}z_1, \dots, t^{\frac{1}{2m_{n-1}}}a_{n-1,n-1}z_{n-1}\right) \\ &= P\left(t^{\frac{1}{2m_1}}a_{1,1}^{-1}z_1, \dots, t^{\frac{1}{2m_{n-1}}}a_{n-1,n-1}^{-1}z_{n-1}\right). \end{aligned}$$

Since P is a weighted homogeneous polynomial with weight $(m_1, \ldots, m_{n-1})$, it follows that

$$
\begin{aligned}
P\left(z_1, \ldots, z_{n-1}\right) &= P\left(a_{1,1} z_1, \ldots, a_{n-1,n-1} z_{n-1}\right) \\
&= P\left(a_{1,1}^{-1} z_1, \ldots, a_{n-1,n-1}^{-1} z_{n-1}\right)
\end{aligned}
$$

for all $z' \in \mathbb{C}^{n-1}$. Therefore, we conclude that

$$\varphi(z) := (a_{1,1} z_1, a_{2,2} z_2, \ldots, a_{n-1,n-1} z_{n-1}, z_n)$$

is an automorphism of M_P, that is, $\varphi \in G_P$. Replacing f by $f \circ \varphi^{-1}$, one may assume that $a_{1,1} = \ldots = a_{n-1,n-1} = 1$. Thus, we obtain $df = \mathrm{Id}$ at the origin.

Case 2 $m_1 \geq m_2 \geq \ldots \geq m_{n-1}$. Following Case 1, one can write $f(z) = (Az' + g(z'), z_n)$, where $g = (g_1, \ldots, g_{n-1})$ is holomorphic in a neighborhood of the origin in $\mathbb{C}^{n-1}$ such that each g_j has weighted order greater than $1/2m_j$, $j = 1, \ldots, n-1$. Collecting the terms of weighted order 1, (7) yields the mapping $(z', z_n) \mapsto (Az', z_n)$ which belongs to G_P. Therefore, after taking a composition with $(z', z_n) \mapsto (A^{-1}z', z_n)$, we may assume that $df = \mathrm{Id}$ at the origin.
Now our goal is to prove that $f = \mathrm{Id}$. Indeed, we may assume that $\tilde{f}(z') = z' + g(z')$, i.e., for each $1 \leq j \leq n-1$,

$$f_j(z') = z_j + g_j(z'),$$

where $g = (g_1, \ldots, g_{n-1})$ is holomorphic in a neighborhood of the origin in $\mathbb{C}^{n-1}$ such that each g_j has weighted order greater than $1/2m_j$, $j = 1, \ldots, n-1$. Therefore, we have

$$P\left(z_1 + g_1(z'), z_2 + g_2(z'), \ldots, z_{n-1} + g_{n-1}(z')\right) = P(z') \tag{12}$$

for all $z' \in U(0)$. Since $\{P = 0\}$ contains no non-trivial analytic set passing through the origin, comparison of the weighted orders of terms in (12) shows that $g_1 \equiv \ldots \equiv g_{n-1} \equiv 0$ on $U(0)$. Hence, by the Identity Theorem, we conclude that $f = \mathrm{Id}$. □

We are now ready to prove Theorem 1.

Proof (Proof of Theorem 1) Let $f \in \mathrm{Aut}(D_P)$ be arbitrary. Then, since $D_P \subset \mathbb{C}^n$ is a bounded pseudoconvex domain of finite type, f extends smoothly to $\overline{D_P}$ (see [3]). Therefore, the points $(0', e^{i\theta})$ are preserved by f. Thus, $f_j(0', z_n) \equiv 0$ for $j = 1, \ldots, n-1$ and $f\,|_{D_P \cap \{z'=0\}} \in \mathrm{Aut}(\Delta)$, where Δ is the unit disc in $\mathbb{C}$. Moreover, it follows that

$$f_n(z) = f_n(0', z_n) = e^{i\theta_n} \frac{z_n - a}{1 - \bar{a} z_n}$$

for some $a \in \Delta$ and $\theta_n \in \mathbb{R}$. Consequently, we have $f(0) = (0', -a)$ (up to a rotation in the z_n-direction). Replacing f by $\phi_{-a,-\theta_n} \circ f$, we may assume that $f(0) = 0$. This yields

$$f_n(z) = e^{i\theta_n} z_n.$$

Moreover, since $f \in \mathrm{Aut}(D_P)$, we get

$$|z_n|^2 + P(f_1(z), \dots, f_{n-1}(z)) \le 1$$

if and only if $|z_n|^2 + P(z') \le 1$. A direct computation together with the invariance of the boundary ∂D_P under biholomorphisms shows that $f_1, \dots, f_{n-1}$ are independent of z_n and holomorphic in a neighborhood of $0 \in \mathbb{C}^{n-1}$. Moreover, we get

$$P(f_1(z'), \dots, f_{n-1}(z')) = P(z')$$

for all z' in a neighborhood of $0 \in \mathbb{C}^{n-1}$. Thus it follows from Lemma 8 that $f \in G_P$ which completes the proof. □

The following theorem is essentially well-known (cf. [2]).

Theorem 5 *Let P be a weighted homogeneous polynomial with weight $(m_1, \dots, m_{n-1})$ given by (6) such that $P(z') > 0$ for all $z' \in \mathbb{C}^{n-1} \setminus \{0\}$. Then, D_P is biholomorphically equivalent to Q_P.*

Now we shall compute the $\mathrm{Aut}(Q_P)$, where

$$Q_P := \{(z', z_n) \in \mathbb{C}^n : \mathrm{Re}\, z_n + P(z') < 0\},$$

where P is given by (6) and $P(z') > 0$ for all $z' \in \mathbb{C}^{n-1} \setminus \{0\}$. We give at first the following lemma which can be derived easily from a straightforward computation.

Proposition 2 *Let P be a weighted homogeneous polynomial with weight $(m_1, \dots, m_{n-1})$ given by (6) such that $P(z') > 0$ for all $z' \in \mathbb{C}^{n-1} \setminus \{0\}$. Then,* $\mathrm{Aut}(Q_P)$ *contains the automorphisms $f_{\alpha,\beta}$, $\alpha > 0$, and $\beta \in \mathbb{R}$, defined by*

$$(z', z_n) \mapsto \left(\frac{(\alpha)^{1/2m_1}}{(1 + i\beta z_n)^{1/m_1}} z_1, \dots, \frac{(\alpha)^{1/2m_{n-1}}}{(1 + i\beta z_n)^{1/m_{n-1}}} z_{n-1}, \frac{\alpha z_n}{1 + i\beta z_n} \right).$$

Conversely, if $\mathrm{Aut}(M_P)$ contains the automorphism $f_{\alpha,\beta}$ for some $\alpha, \beta \in \mathbb{R}$ with $\alpha > 0$ and $\beta \neq 0$, then M_P is exactly Q_P. More precisely, we have the following proposition.

Proposition 3 *Let P be a weighted homogeneous polynomial with weight $(m_1, \dots, m_{n-1})$ given by (1) such that $P(z') > 0$ for all $z' \in \mathbb{C}^{n-1} \setminus \{0\}$. Suppose that* $\mathrm{Aut}(M_P)$ *contains the following automorphisms $f_{\alpha,\beta}$ defined by*

$$(z', z_n) \mapsto \left(\frac{(\alpha)^{1/2m_1}}{(1 + i\beta z_n)^{1/m_1}} z_1, \dots, \frac{(\alpha)^{1/2m_{n-1}}}{(1 + i\beta z_n)^{1/m_{n-1}}} z_{n-1}, \frac{\alpha z_n}{1 + i\beta z_n} \right)$$

for some $\alpha, \beta \in \mathbb{R}$ with $\alpha > 0$ and $\beta \neq 0$. Then, the polynomial P always has the following form:

$$P(z') = \sum_{wt(K) = wt(L) = 1/2} a_{KL} {z'}^K \bar{z'}^L,$$

where $a_{KL} \in \mathbb{C}$ *with* $a_{KL} = \bar{a}_{LK}$.

Proof Since $f_{\alpha,\beta} \in \mathrm{Aut}(M_P)$ for some $\alpha, \beta \in \mathbb{R}$ with $\alpha > 0$ and $\beta \neq 0$, it follows that

$$\mathrm{Re}\frac{z_n}{1+i\beta z_n} + P\left(\frac{1}{(1+i\beta z_n)^{1/m_1}}z_1, \ldots, \frac{1}{(1+i\beta z_n)^{1/m_{n-1}}}z_{n-1}\right) = 0$$

for all $z \in \partial M_P$. This is equivalent to

$$\mathrm{Re}\left(z_n - i\beta z_n^2 + \ldots\right) + P\left(z_1 - \frac{i\beta z_n z_1}{m_1} + \ldots, \ldots, z_{n-1} - \frac{i\beta z_n z_{n-1}}{m_{n-1}} + \ldots\right) = 0$$

for all $z \in \partial M_P$, where the dots denote terms of weight greater than 2. By expanding P into Taylor series, one has

$$\mathrm{Re}\ z_n + \mathrm{Re}\left(-i\beta z_n^2\right) + P(z') + \mathrm{Re}\left(-i\beta z_n \sum_{j=1}^{n-1} \frac{\partial P}{\partial z_j}(z')\frac{z_j}{m_j}\right) + \ldots = 0$$

for all $z \in \partial M_P$, where the dots denote terms of weight greater than 2. Therefore, we obtain

$$\mathrm{Re}\left(-i\beta z_n^2\right) + \mathrm{Re}\left(-i\beta z_n \sum_{j=1}^{n-1} \frac{\partial P}{\partial z_j}(z')\frac{z_j}{m_j}\right) = 0$$

for all $z \in \partial M_P$. Moreover, if we let $z_n = -P(z')$, then we have

$$\mathrm{Re}\left(i \sum_{j=1}^{n-1} \frac{\partial P}{\partial z_j}(z')\frac{z_j}{m_j}\right) = 0$$

for all $z' \in \mathbb{C}^{n-1}$. In conclusion, Lemma 4 ensures that

$$P(z') = \sum_{wt(K)=wt(L)=1/2} a_{KL} {z'}^K \bar{z'}^L,$$

where $a_{KL} \in \mathbb{C}$ with $a_{KL} = \bar{a}_{LK}$. □

5 Automorphisms of a Finite Multitype Model

In this section, we provide the proof of Theorem 2 as our second main result. First of all, we recall some notations and definitions. Let S_λ $(\lambda > 0)$, $T_s (s \in \mathbb{R})$ be automorphisms of M_P which are defined, respectively, by

$$S_\lambda(z) = (\lambda^{1/2m_1}z_1, \ldots, \lambda^{1/2m_{n-1}}z_{n-1}, \lambda z_n);\ \ T_s(z) = (z', z_n + is).$$

Definition 2 A model M_P is called *tubular* (resp. *rotational*) if M_P is biholomorphically equivalent to a model $M_{\widetilde{P}}$, where a weighted homogeneous polynomial $\widetilde{P}$ satisfies $\widetilde{P}(z_1, \ldots, z_{n-1}) = \widetilde{P}(\text{Im } z_1, z_2, \ldots, z_{n-1})$ (resp. $\widetilde{P}(z_1, \ldots, z_{n-1}) = \widetilde{P}(|z_1|, z_2, \ldots, z_{n-1})$) for all $z' \in \mathbb{C}^{n-1}$.

Definition 3 A model M_P is called *generic* if it is not biholomorphically equivalent to any rotational model or to any tubular model.

By expanding P into Taylor series at $\alpha = (\alpha_1, \ldots, \alpha_{n-1}) \in \mathbb{C}^{n-1}$, one has

$$
\begin{aligned}
P(z') &= \sum_{wt(K)+wt(L)=1} a_{KL} z'^K \bar{z}'^L \\
&= P(\alpha) + 2\text{Re} \sum_{|p|>0} \frac{D^p P(\alpha)}{p!}(z' - \alpha)^p + \sum_{|p|,|q|>0} \frac{D^p \overline{D}^q P(\alpha)}{p!q!}(z' - \alpha)^p (\bar{z}' - \bar{\alpha})^q,
\end{aligned}
$$

where D^p and $\overline{D}^q$ denote the partial differential operators

$$
\frac{\partial^{|p|}}{\partial z_1^{p_1} \ldots \partial z_{n-1}^{p_{n-1}}} \text{ and } \frac{\partial^{|q|}}{\partial \bar{z}_1^{q_1} \ldots \partial \bar{z}_{n-1}^{q_{n-1}}},
$$

respectively. By the following change of variables

$$
\begin{cases}
w_n = z_n + P(\alpha) + 2\sum_{|p|>0} \frac{D^p P(\alpha)}{p!}(z' - \alpha)^p \\
w' = z' - \alpha,
\end{cases}
$$

a defining function for M_P is now given by

$$
\begin{aligned}
\rho(z) &= \text{Re } w_n + \sum_{|p|,|q|>0} \frac{D^p \overline{D}^q P(\alpha)}{p!q!}(w')^p (\bar{w}')^q \\
&= \text{Re } w_n + \sum_{|p|,|q|>0;\ wt(p)+wt(q)<1} \frac{D^p \overline{D}^q P(\alpha)}{p!q!}(w')^p (\bar{w}')^q \\
&\quad + \sum_{|p|,|q|>0;\ wt(p)+wt(q)=1} \frac{D^p \overline{D}^q P(\alpha)}{p!q!}(w')^p (\bar{w}')^q.
\end{aligned}
$$

In what follows, we assume that M_P is generic. Moreover, we introduce the notation

$$
P_{2m_1,\ldots,2m_{n-1}}(\partial M_P) := \{z \in \partial M_P : \mathcal{M}(z) = (2m_1, 2m_2, \ldots, 2m_{n-1}, 1)\}
$$

and $\Gamma := \{(0', it) : t \in \mathbb{R}\}$.

Lemma 9 *Let P be a weighted homogeneous polynomial with weight $(m_1, \dots, m_{n-1})$ given by (1) such that $P(z') > 0$ for all $z' \in \mathbb{C}^{n-1} \setminus \{0\}$. Suppose that M_P is generic. If at least one of the integers $m_1, \dots, m_{n-1}$ is greater than 2, then*

$$P_{2m_1,\dots,2m_{n-1}}(\partial M_P) = \Gamma := \{(0', it) \colon t \in \mathbb{R}\}.$$

Proof It is easy to show that $\Gamma \subset P_{2m_1,\dots,2m_{n-1}}(\partial M_P)$. So, it suffices to show that $P_{2m_1,\dots,2m_{n-1}}(\partial M_P) \subset \Gamma$. Let $p = (\alpha, -P(\alpha) + it)$ $(\alpha = (\alpha_1, \dots, \alpha_{n-1}) \neq 0)$ be any boundary point in $\partial M_P \setminus \Gamma$.

Note that by [6, Main Theorem, p. 531], we have

$$\mathcal{M}(p) \leq (2m_1, \dots, 2m_{n-1}, 1).$$

Therefore, if $\mathcal{M}(p) = (2m_1, \dots, 2m_{n-1}, 1)$, then

$$D^p \overline{D}^q P(\alpha) = 0 \text{ whenever } wt(p) + wt(q) < 1.$$

Hence, we obtain

$$\begin{aligned} P(\alpha + z') &= P(\alpha) + 2\mathrm{Re} \sum_{|p|>0;\ wt(p)\leq 1} \frac{D^p P(\alpha)}{p!}(z')^p \\ &\quad + \sum_{|p|,|q|>0;\ wt(p)+wt(q)=1} \frac{D^p \overline{D}^q P(\alpha)}{p!q!}(z')^p(\bar{z}')^q. \end{aligned}$$

This implies that

$$P_{j,\bar{k}}(\alpha + z') = P_{j,\bar{k}}(z'),\ \ j, k = 1, \dots, n-1, \tag{13}$$

where $P_{j,\bar{k}}(z') = \frac{\partial^2 P}{\partial z_j \partial \bar{z}_k}(z')$. By a change of coordinates, we may assume that $\alpha = (1, 0, \dots, 0)$. Fix z_ℓ for all $\ell \geq 2$ and let

$$f(x, y) = P_{1,\bar{1}}(x + iy, z_2, \dots, z_{n-1})$$

for all $z_1 := x + iy \in \mathbb{C}$. Thus, it follows from (13) that $f(x+1, y) = f(x, y)$ for all $(x, y) \in \mathbb{R}^2$. Hence, for each $y \in \mathbb{R}$ $f(x, y)$ is a periodic polynomial in x, and thus $f(x, y)$ does not depend on x, i.e., $f(x, y) = g(y)$, where g is a polynomial in y. Combining this fact with the assumption that P has no harmonic terms, one can conclude that $P(z_1, \dots, z_{n-1}) = P(\mathrm{Im}\, z_1, z_2, \dots, z_{n-1})$ for all $z' \in \mathbb{C}^{n-1}$, and hence M_P is biholomorphically equivalent to a tubular model. This leads to a contradiction and hence the proof is complete. □

We now prepare the following theorem as one of the main ingredients in proving Theorem 2.

Theorem 6 *Let P be a weighted homogeneous polynomial with weight $(m_1, \ldots, m_{n-1})$ given by (1) such that $P(z') > 0$ for all $z' \in \mathbb{C}^{n-1} \setminus \{0\}$. Suppose that M_P is a generic model which is not biholomorphically equivalent to Q_P. Suppose that $f \in \mathrm{Aut}(M_P)$, $f(0) = 0$ and there exist neighborhoods U_1, U_2 of $0 \in \mathbb{C}^n$ such that f extends to a local diffeomorphism between $U_1 \cap \overline{M_P}$ and $U_2 \cap \overline{M_P}$. Then after compositions with S_t $(t > 0)$ or with an element of G_P if necessary, $f = \mathrm{Id}$.*

Proof Let us define a set $\mathcal{H}$ by setting $\mathcal{H} := \{z \in \mathbb{C} : \mathrm{Re}\, z < 0\}$ and recall that $\Gamma := \{(0', it) : t \in \mathbb{R}\}$. Then we consecutively define $g_j(z_n) := f_j(0', z_n)$ $(1 \le j \le n-1)$, and $g_n(z_n) := f_n(0', z_n)$ for all $z_n \in \mathcal{H}$. Since the Catlin's multitype is a CR-invariant, it follows from Lemma 9 that, after shrinking the neighborhoods U_1, U_2 if necessary, we may assume that $f(U_1 \cap \Gamma) = U_2 \cap \Gamma$. Consequently, for each $1 \le j \le n-1$, we have $g_j(it) = 0$ for all $-\epsilon_0 < t < \epsilon_0$ with $\epsilon_0 > 0$ small enough. Then it follows from the Identity Theorem that $g_j(z_n) = 0$ for all $z_n \in \mathcal{H}$. Moreover, since $P(z') > 0$ for all $z' \in \mathbb{C}^{n-1} \setminus \{0\}$, we have $g_n \in \mathrm{Aut}(\mathcal{H})$. Since $g_n(0) = 0$, one can show that $g_n(z_n) = \dfrac{\alpha z_n}{1 + i\beta z_n}$ for some $\alpha > 0$ and $\beta \in \mathbb{R}$. In addition, since $f_n(M_P) \subset \mathcal{H}$ and f is biholomorphic, we immediately obtain $f_n(z) = f_n(0', z_n) = \dfrac{\alpha z_n}{1 + i\beta z_n}$ for some $\alpha > 0$ and $\beta \in \mathbb{R}$.

We now consider the following cases:

Case 1 $\beta \neq 0$.

In this case, by expanding f_n into Taylor series, one can obtain

$$f_n(z) = \frac{\alpha z_n}{1 + i\beta z_n} = \alpha z_n - i\beta\alpha z_n^2 + \ldots,$$

where the dots denote terms of weight greater than 2. Moreover, due to the invariance of $\overline{M_P}$ under any map S_t $(t > 0)$, we get

$$\begin{aligned} \mathrm{Re}\left(f_n\left(t^{\frac{1}{2m_1}} z_1, \ldots, t^{\frac{1}{2m_{n-1}}} z_{n-1}, t z_n\right)\right) \\ + P\left(f_1\left(t^{\frac{1}{2m_1}} z_1, \ldots, t^{\frac{1}{2m_{n-1}}} z_{n-1}, t z_n\right), \ldots, f_{n-1} \right. \\ \left. \times \left(t^{\frac{1}{2m_1}} z_1, \ldots, t^{\frac{1}{2m_{n-1}}} z_{n-1}, t z_n\right)\right) \le 0 \end{aligned} \tag{14}$$

for all $(z', z_n) \in U_1 \cap \overline{M_P}$ and $t \in (0, 1)$. Therefore, (14) is equivalent to

$$\begin{aligned} &\mathrm{Re}\big(\alpha t z_n - i\beta\alpha t^2 z_n^2 + o(t^2)\big) \\ &+ P\big(f_1(t^{\frac{1}{2m_1}} z_1, \ldots, t^{\frac{1}{2m_{n-1}}} z_{n-1}, t z_n), \ldots, f_{n-1}(t^{\frac{1}{2m_1}} z_1, \ldots, t^{\frac{1}{2m_{n-1}}} z_{n-1}, t z_n)\big) \le 0 \end{aligned}$$

for all $(z', z_n) \in U_1 \cap \overline{M_P}$ and $t \in (0, 1)$. Without loss of generality, we may assume that

$$m_1 \ge m_2 \ge \ldots \ge m_{n-1}.$$

In what follows, denote by $h_s(z)$ a germ at the origin of holomorphic functions with weighted order greater than s ($s > 0$).

We shall prove that $df = \mathrm{Id}$ at the origin, up to a composition with an element of G_P. To prove this, we divide the argument into the following two sub-cases:

Sub-case 1 $m_1 > m_2 > \ldots > m_{n-1}$. Fix a point $z \in U_1 \cap \partial M_P$. Then, since $\mathrm{Re}\left(f_n\left(t^{\frac{1}{2m_1}} z_1, \ldots, t^{\frac{1}{2m_{n-1}}} z_{n-1}, tz_n\right)\right) = \alpha t \mathrm{Re}\, z_n + o(t)$, it follows that

$$P\left(f_1\left(t^{\frac{1}{2m_1}} z_1, \ldots, t^{\frac{1}{2m_{n-1}}} z_{n-1}, tz_n\right), \ldots, f_{n-1}\left(t^{\frac{1}{2m_1}} z_1, \ldots, t^{\frac{1}{2m_{n-1}}} z_{n-1}, tz_n\right)\right)$$
$$= -\alpha t \mathrm{Re}\, z_n + o(t).$$

Moreover, since $t^{\frac{1}{2m_1}} > t^{\frac{1}{2m_2}} > \ldots > t^{\frac{1}{2m_{n-1}}}$ for any $t \in (0, 1)$, one has for each $1 \le j \le n-1$

$$f_j(z) = a_{j,j} z_j + h_{1/2m_j}(z),$$

where $a_{1,1}, \ldots, a_{n-1,n-1} \ne 0$.

Next, replacing f by $S_{1/\alpha} \circ f$, we may assume that $\alpha = 1$. Taking the first-order partial derivative of both sides of the inequality (14) with respect to t and then evaluating its limit as $t \to 0^+$, we arrive at

$$\mathrm{Re}\, z_n + P(a_{1,1}z_1, a_{2,2}z_2, \ldots, a_{n-1,n-1}z_{n-1}) < 0$$

for all $(z', z_n) \in M_P$. A similar argument for f^{-1} gives

$$\mathrm{Re}\, z_n + P(a_{1,1}^{-1}z_1, a_{2,2}^{-1}z_2, \ldots, a_{n-1,n-1}^{-1}z_{n-1}) < 0$$

for all $(z', z_n) \in M_P$. Altogether, we conclude that

$$\mathrm{Re}\, z_n + P(a_{1,1}z_1, a_{2,2}z_2, \ldots, a_{n-1,n-1}z_{n-1}) < 0$$

if and only if $\mathrm{Re}\, z_n + P(z') < 0$, and hence

$$g(z) := (a_{1,1}z_1, a_{2,2}z_2, \ldots, a_{n-1,n-1}z_{n-1}, z_n)$$

is an automorphism of M_P, that is, $g \in G_P$. Replacing f by $f \circ g^{-1}$, one may assume that $a_{1,1} = \ldots = a_{n-1,n-1} = 1$. Thus, we obtain $df = \mathrm{Id}$ at the origin.

Sub-case 2 $m_1 \ge m_2 \ge \ldots \ge m_{n-1}$. Following Sub-case 1, one can write $f(z) = (Az' + g(z), z_n)$, where $g = (g_1, \ldots, g_{n-1})$ is holomorphic in a neighborhood of the origin in $\mathbb{C}^n$ such that each g_j has weighted order greater than $1/2m_j$, $j = 1, \ldots, n-1$. Collecting the terms of weighted order 1, (14) yields the mapping $(z', z_n) \mapsto (Az', z_n)$ which belongs to G_P. Therefore, after taking a composition with $(z', z_n) \mapsto (A^{-1}z', z_n)$, we may assume that $df = \mathrm{Id}$ at the origin.

Now our goal is to prove that $f = \mathrm{Id}$. Aiming for a contradiction, suppose otherwise that $f \neq \mathrm{Id}$. We may assume that $f(z) = z + g(z)$, i.e., for each $1 \leq j \leq n-1$,

$$f_j = z_j + g_j(z),$$

where $g = (g_1, \ldots, g_{n-1})$ is holomorphic in a neighborhood of the origin in $\mathbb{C}^n$ such that each g_j has weighted order greater than $1/2m_j$, $j = 1, \ldots, n-1$. Therefore, we have

$$\mathrm{Re}\left(z_n - i\beta z_n^2 + \ldots\right) + P\left(z_1 + g_1(z), z_2 + g_2(z), \ldots, z_{n-1} + g_{n-1}(z)\right) = 0$$

for all $z \in U_1 \cap \partial M_P$, or equivalently

$$\mathrm{Re}\; z_n + \mathrm{Re}\left(-i\beta z_n^2\right) + P(z') + 2\mathrm{Re}\left(\sum_{j=1}^{n-1} \frac{\partial P}{\partial z_j}(z') g_j(z)\right) + h_2(z) = 0$$

for all $z \in U_1 \cap \partial M_P$. This implies that

$$\mathrm{Re}\left(-i\beta z_n^2\right) + 2\mathrm{Re}\left(\sum_{j=1}^{n-1} \frac{\partial P}{\partial z_j}(z') g_j(z)\right) + h_2(z) = 0 \tag{15}$$

for all $z \in U_1 \cap \partial M_P$. (Here, we recall that $h_2(z)$ is a germ at the origin of holomorphic functions with weighted order greater than 2.)

Now if we set $z_n = -P(z') + it$ for $t \in \mathbb{R}$, then $z_n^2 = P^2(z') - t^2 - 2itP(z')$, and hence $\mathrm{Re}\left(-i\beta z_n^2\right) = -2\beta t P(z')$. Substituting this into (15), we obtain

$$-2\beta t P(z') + 2\mathrm{Re}\left(\sum_{j=1}^{n-1} \frac{\partial P}{\partial z_j}(z') g_j(z', -P(z') + it)\right) + h_2(z) = 0. \tag{16}$$

Setting the coefficients of t^k in (16) equal zero for $k \in \mathbb{N}$, we conclude that $g_j(z) = a_j z_n z_j + \ldots$ for $j = 1, \ldots, n-1$, where the dots indicate terms of higher weight. Differentiating the terms of weighted order 1 in (16) with respect to t and then setting $t = 0$, one gets

$$P(z') = \frac{1}{\beta}\mathrm{Re}\left(\sum_{j=1}^{n-1} \frac{\partial P}{\partial z_j}(z') i a_j z_j\right).$$

Therefore, according to Lemma 3, we should have $a_j = -i\beta/m_j$ for $j = 1, \ldots, n-1$. Collecting the terms of weighted order 1 in (16) at $t = 0$ and then utilizing Lemma 4, we have

$$P(z') = \sum_{wt(K)=wt(L)=1/2} a_{KL} z'^K \bar{z}'^L,$$

where $a_{KL} \in \mathbb{C}$ with $a_{KL} = \bar{a}_{LK}$. Therefore, if $\beta \neq 0$, then M_P is biholomorphically equivalent to Q_P, which leads to a contradiction.

Case 2 $\beta = 0$. In this case we immediately obtain $f_n(z) = \alpha z_n$ for some $\alpha > 0$. Without loss of generality, we may assume that $\alpha = 1$. Since f can be smoothly extended to the boundary of M_P (cf. [3]), we obtain

$$\text{Re } z_n + P(f_1(z), \ldots, f_{n-1}(z)) \leq 0$$

if and only if Re $z_n + P(z') \leq 0$. We note that $f_1, \ldots, f_{n-1}$ are independent of the variable z_n due to the invariance of the boundary under the actions of automorphism group. Furthermore, by Proposition 1, $f_1, \ldots, f_{n-1}$ can be extended to holomorphic functions in a neighborhood of $0 \in \mathbb{C}^{n-1}$. This yields

$$P(f_1(z'), \ldots, f_{n-1}(z')) = P(z')$$

for all z' in a neighborhood of $0 \in \mathbb{C}^{n-1}$. Then it follows from Lemma 8 that $f \in G_P$, and thus the proof is complete. □

Now we are ready to prove Theorem 2.

Proof (Proof of Theorem 2) Let $f \in \text{Aut}(M_P)$ be arbitrary. Then, by Proposition 1, it follows that there exist $p \in \Gamma$ and $q \in \Gamma$ such that f and f^{-1} extend to be holomorphic in neighborhoods of p and q, respectively, and $f(p) = q$. Replacing f by its composition with reasonable translations T_t, we may assume that $p = q = (0, 0)$, and there exist neighborhoods U_1 and U_2 of $(0, 0)$ such that $U_2 \cap \partial M_P = f(U_1 \cap \partial M_P)$, and f and f^{-1} are holomorphic in U_1 and U_2, respectively. Moreover, f is a local CR diffeomorphism between $U_1 \cap \partial M_P$ and $U_2 \cap \partial M_P$. Therefore, the assertion follows from Theorem 6. □

We close this paper by exploring several known examples through our main theorems.

Example 1 Let $E_{1,m}$ be the ellipsoid

$$E_{1,m} := \{(z_1, z_2) \in \mathbb{C}^2 \colon |z_2|^2 + |z_1|^{2m} < 1\}, \ m \geq 2.$$

For the ellipsoid $E_{1,m}$, the polynomial P is given by $P(z_1) = |z_1|^{2m}$. Then $P(f_1(z_1)) \equiv P(z_1)$ if and only if $f_1(z_1) = e^{i\theta} z_1$ for some $\theta \in \mathbb{R}$. Therefore, from Theorem 1 we conclude that

$$\text{Aut}(E_{1,m}) = \left\{(z_1, z_2) \mapsto \left(e^{i\theta_1} \frac{(1-|a|^2)^{1/2m}}{(1-\bar{a}z_2)^{1/m}} z_1, e^{i\theta_2} \frac{z_2 - a}{1 - \bar{a}z_2}\right) : |a| < 1, \theta_1, \theta_2 \in \mathbb{R}\right\},$$

which is already well-known.

Example 2 Consider the domain

$$\Omega := \{(z_1, z_2, z_3) \in \mathbb{C}^3 \colon |z_3|^2 + |z_1|^4 + |z_2|^4 + (\bar{z}_2 z_1 + \bar{z}_1 z_2)^2 < 1\}.$$

In this case, the polynomial P is given by $P(z_1, z_2) = |z_1|^4 + |z_2|^4 + (\bar{z}_2 z_1 + \bar{z}_1 z_2)^2$. Then a direct computation shows that $P(Az') \equiv P(z')$ if and only if $Az' = e^{i\theta}(z_2, z_1)$

or $Az' = e^{i\theta}(z_1, z_2)$ for some $\theta \in \mathbb{R}$. Hence, it follows from Theorem 1 that $\mathrm{Aut}(\Omega)$ is generated by

$$(z_1, z_2, z_3) \mapsto \left(\frac{(1-|a|^2)^{1/4}}{(1-\bar{a}z_3)^{1/2}} z_1, \frac{(1-|a|^2)^{1/4}}{(1-\bar{a}z_3)^{1/2}} z_2, \frac{z_3 - a}{1-\bar{a}z_3} \right)$$

and

$$(z_1, z_2, z_3) \mapsto \left(e^{i\theta_1} z_{\sigma(1)}, e^{i\theta_1} z_{\sigma(2)}, e^{i\theta_2} z_3 \right),$$

where $a \in \Delta, \theta_1, \theta_2 \in \mathbb{R}$, and σ is a permutation of the set $\{1, 2\}$. This result is already proved in [9].

Example 3 Let Ω_{HKN} be the Kohn–Nirenberg domain, introduced first in [15] and defined by

$$\Omega_{HKN} := \left\{ (z, w) \in \mathbb{C}^2 \colon \mathrm{Re}\, w + |z|^8 + \frac{15}{7} |z|^2 \mathrm{Re}(z^6) < 0 \right\}.$$

In this case, the polynomial P is given by $P(z) = |z|^8 + \frac{15}{7}|z|^2\mathrm{Re}(z^6)$. We see that P is homogeneous of degree 8 and $P(f(z)) \equiv P(z)$ if and only if $f(z) = e^{k\pi i/3} z$ for $k \in \{0, 1, \ldots, 5\}$. Therefore, from Theorem 2 we have

$$\mathrm{Aut}(\Omega_{HKN}) = \left\{ (z, w) \mapsto \left(\sqrt[8]{\lambda} e^{k\pi i/3} z, \lambda w + it \right) : k = 0, \ldots, 5; t \in \mathbb{R}, \lambda > 0 \right\},$$

as shown in [19, Theorem 2].

Example 4 Let E be the ellipsoid

$$E := \{(z_1, z_2, z_3) \in \mathbb{C}^3 \colon |z_3|^2 + |z_1|^4 + |z_2|^6 < 1\}.$$

For the ellipsoid E, the polynomial P is given by $P(z_1, z_2) = |z_1|^4 + |z_2|^6$. Then $P(f_1(z_1, z_2), f_2(z_1, z_2)) \equiv P(z_1, z_2)$ if and only if $f_1(z_1) = e^{i\theta_1} z_1,\ f_2(z_2) = e^{i\theta_2} z_2$ for some $\theta_1, \theta_2 \in \mathbb{R}$. Therefore, from Theorem 1 we conclude that $\mathrm{Aut}(E)$ includes

$$(z_1, z_2, z_3) \mapsto \left(e^{i\theta_1} \frac{(1-|a|^2)^{1/4}}{(1-\bar{a}z_3)^{1/2}} z_1, e^{i\theta_2} \frac{(1-|a|^2)^{1/6}}{(1-\bar{a}z_3)^{1/3}} z_2, e^{i\theta_3} \frac{z_3 - a}{1-\bar{a}z_3} \right),$$

where $|a| < 1,\ \theta_1, \theta_2, \theta_3 \in \mathbb{R}$.

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