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Calculating Parametric Oscillation of Third-Order Nonlinear System With Dynamic Friction and Fractional Damping

In this paper, the parametric resonance of third-order parametric nonlinear system with dynamic friction and fractional damping is investigated using the asymptotic method. The approximately analytical solution for the system is first determined, and the amplitude–frequency equation of the oscillator is established. The stability condition of the resonance solution is then obtained by means of Lyapunov theory. Additionally, the effect of the fractional derivative on the system dynamics is analyzed. The effects of the two parameters of the fractional-order derivative, i.e., the fractional coefficient and the fractional order, on the amplitude–frequency curves are investigated.
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1 Introduction

Fractional-order calculus includes the operators of fractional-order derivative and fractional-order integral, which is a generalization of the traditional integer-order calculus to the fractional-order and/or complex-order counterpart [1–4]. The most frequently encountered definition of the fractional differential operator is the definition of the Riemann–Liouville operator. In this paper, we use the definition of the Riemann–Liouville fractional differential operator. The applications of fractional calculus in engineering and physics have attracted lots of attention [5–8].

In order to study the periodic oscillation of nonlinear systems, some classical analytical methods, including their improved version, such as harmonic balance method, multiscale method, perturbation method, averaging method, and Kryloff–Bogoliubov–Mitropolskii (KBM) method, are used [9–12]. These methods may be applied to find periodic solutions of the systems having the fractional-order derivative. In recent years, the study of nonlinear oscillations of Duffing systems [13–21], van der Pol systems [22–25], and Mathieu equation [26–28] has been studied extensively by the averaging method, and the asymptotic method. Compared to the traditional integer-order systems, the fractional-order system has the advantage that it describes much closer to the real nature of the world.

The theory of the parametric oscillation of the second-order system has been investigated in a lot of publications [1–4]. In the late 20th century, the vibrations of the third-order system and higher order systems were studied by Dao [29–33]. The parametric resonance oscillations of the third-order nonlinear systems have been studied in detail by the author. In Ref. [32], N.V. Dao has been investigated the influence of the Coulomb friction and of turbulent friction on the parametric oscillation. The author has obtained the amplitude–frequency curves with different values of the friction coefficients.

Based on the results of the paper [32], the parametric oscillation of third-order nonlinear system with dynamic friction and fractional damping is analytically studied by the asymptotic method in this paper. To calculate the oscillation of the system with fractional-order derivative in the Appendix, we have introduced an algorithm to calculate the fractional-order derivative of trigonometric functions $a(t)\cos[\Omega t+\psi(t)]$ and $a(t)\sin[\Omega t+\psi(t)]$ when $a(t)$ and $\psi(t)$ change slowly.

This study is organized in five sections. Section 2 presented the way to find the approximate solution for harmonic resonance of the third-order nonlinear system. Based on Lyapunov theory, the existence condition in harmonic resonance and stability condition for steady-state solution is mentioned in Sec. 3. The influences of the fractional-order parameters on the existence condition in harmonic resonance and on the stability condition for steady-state solution are also analyzed. In Sec. 4, the influences of the fractional-order parameter on the existence condition in harmonic resonance, the steady-state amplitude, the amplitude–frequency curves, and the system stability are studied by the numerical simulation. A comparison between the integer-order and the fractional-order systems is also made in this section [34,35]. Section 5 includes some concluding remarks of this study.

2 Construction of Approximate Solution Using the Asymptotic Method

Let us consider the parametric oscillation of third-order nonlinear system with dynamic friction and fractional damping governed by differential equation

$$\dddot{x}+\alpha\ddot{x}+\omega^2\dot{x}+\alpha\omega^2 x+\varepsilon[kx^3+h\dot{x}^3+h_2\dot{x}^2\operatorname{sign}\dot{x}+\delta_p D^p x - \mathrm{c}x\cos\Omega t]=0 \quad (1)$$

where $\alpha,\omega,k,h,\delta_p,c,\Omega$ are constants, h_2 is positive constant, ε is a small parameter, $D^p x$ is p-order fractional derivative of $x(t)$, is the function characterizing the nonlinear friction.

It is supposed that a parametric resonance relation is given as

$$\varepsilon\sigma=\omega^2(1-\eta^2),\quad \eta=\frac{\Omega}{2\omega} \quad (2)$$

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Then Eq. (1) can be rewritten in the following form:

$$\dddot{x} + \alpha\ddot{x} + \frac{\Omega^2}{4}\dot{x} + \alpha\frac{\Omega^2}{4}x + \varepsilon f(x,\dot{x},\ddot{x},D^p x) - \varepsilon c x\cos\Omega t = 0 \quad (3)$$

where

$$f(x,\dot{x},\ddot{x},D^p x) = \sigma\dot{x} + \alpha\sigma x + kx^3 + h\dot{x}^3 + h_2\dot{x}^2\,\mathrm{sign}\,\dot{x} + \delta_p D^p x \quad (4)$$

Using the asymptotic method, the periodic partial solution of Eq. (3) can be found under the series

$$x = a\cos\left(\frac{\Omega}{2}t+\psi\right) + \varepsilon u_1\left(a,\psi,\frac{\Omega}{2}t\right) + \varepsilon^2 u_2\left(a,\psi,\frac{\Omega}{2}t\right) + \cdots \quad (5)$$

In which $u_i(a,\psi,\theta)$ $(i=1,2,\ldots)$ is the periodic function of ψ and θ, $(\theta = \frac{\Omega t}{2})$, with period 2π, and functions $a(t)$ and $\psi(t)$ are determined from the following equations:

$$\begin{aligned}\frac{da}{dt} &= \varepsilon A_1(a,\psi) + \varepsilon^2 A_2(a,\psi) + \cdots \\ \frac{d\psi}{dt} &= \varepsilon B_1(a,\psi) + \varepsilon^2 B_2(a,\psi) + \cdots\end{aligned} \quad (6)$$

To determine the unknown functions u_s, A_s, B_s, the following derivatives are calculated as:

$$\frac{dx}{dt} = -\frac{\Omega}{2}a\sin\varphi + \varepsilon\left(A_1\cos\varphi - aB_1\sin\varphi + \frac{\partial u_1}{\partial t}\right) + \varepsilon^2\ldots \quad (7)$$

$$\frac{d^2x}{dt^2} = -\frac{\Omega^2}{4}a\cos\varphi + \varepsilon\left(-\Omega A_1\sin\varphi - \Omega aB_1\cos\varphi + \frac{\partial^2 u_1}{\partial t^2}\right) + \varepsilon^2\ldots \quad (8)$$

$$\frac{d^3x}{dt^3} = \frac{\Omega^3}{8}a\sin\varphi + \varepsilon\left(-\frac{3}{4}\Omega^2 A_1\cos\varphi + \frac{3}{4}\Omega^2 aB_1\sin\varphi + \frac{\partial^3 u_1}{\partial t^3}\right) + \varepsilon^2\ldots \quad (9)$$

where

$$\varphi = \frac{\Omega}{2}t + \psi \quad (10)$$

Substituting Eqs. (5), (7), (8), and (9) into Eq. (3) and comparing the coefficients of the same degree ε on both sides, we obtain

$$\begin{aligned}&\frac{\partial u_1^3}{\partial t^3} + \alpha\frac{\partial u_1^2}{\partial t^2} + \frac{\Omega^2}{4}\frac{\partial u_1}{\partial t} + \alpha\frac{\Omega^2}{4}u_1 - \left(\frac{\Omega^2}{2}A_1 + \Omega\alpha aB_1\right)\cos\varphi \\ &+\left(\Omega^2\frac{a}{2}B_1 - \Omega\alpha A_1\right)\sin\varphi = -f_0 + ac\cos\varphi\cos\Omega t\end{aligned} \quad (11)$$

where

$$\begin{aligned}f_0 &= f\left[a\cos\varphi, -\frac{\Omega}{2}a\sin\varphi, -\frac{\Omega^2}{4}a\cos\varphi, D^p(a\cos\varphi)\right] \\ &= -\sigma\frac{\Omega}{2}a\sin\varphi + \alpha\sigma a\cos\varphi + ka^3\cos^3\varphi - h\frac{\Omega^3}{8}a^3\sin^3\varphi \\ &+ h_2\left(-\frac{\Omega}{2}a\sin\varphi\right)^2\mathrm{sign}\left(-\frac{\Omega}{2}a\sin\varphi\right) + \delta_p D^p(a\cos\varphi)\end{aligned} \quad (12)$$

From the Appendix it, is the following expressions:

$$\begin{aligned}D_t^p[a(t)\cos(\Omega t+\psi(t))] &= \Omega^p a(t)\cos\left[\Omega t + \psi(t) + p\frac{\pi}{2}\right] + \varepsilon[\cdot] \\ D_t^p[a(t)\sin(\Omega t+\psi(t))] &= \Omega^p a(t)\sin\left[\Omega t + \psi(t) + p\frac{\pi}{2}\right] + \varepsilon[\cdot]\end{aligned} \quad (13)$$

The right side of Eq. (12) is now rewritten in the form

$$\begin{aligned}f_0 = &-\sigma\frac{\Omega}{2}a\sin\varphi + \alpha\sigma a\cos\varphi + ka^3\cos^3\varphi - h\frac{\Omega^3}{8}a^3\sin^3\varphi \\ &+ h_2\frac{\Omega^2}{4}a^2\sin^2\varphi\,\mathrm{sign}\left(-\frac{\Omega}{2}a\sin\varphi\right) \\ &+ a\delta_p\left(\frac{\Omega}{2}\right)^p\left[\cos\frac{p\pi}{2}\cos\varphi - \sin\frac{p\pi}{2}\sin\varphi\right]\end{aligned} \quad (14)$$

The function f_0 in Eq. (14) is expanded in Fourier series as

$$f_0 = \sum_{m=0}^{\infty}(r_m(a)\cos m\varphi + s_m(a)\sin m\varphi) \quad (15)$$

where

$$\begin{aligned}r_0 &= \frac{1}{2\pi}\int_0^{2\pi} f_0 d\varphi = \langle f_0\rangle \\ r_m &= \frac{1}{\pi}\int_0^{2\pi} f_0\cos m\varphi d\varphi = 2\langle f_0\cos m\varphi\rangle \\ s_m &= \frac{1}{\pi}\int_0^{2\pi} f_0\sin m\varphi d\varphi = 2\langle f_0\sin m\varphi\rangle\end{aligned} \quad (16)$$

in which $\langle F\rangle$ is the operator of the averaging function F on time.

Similarly, function u_1 in Eq. (11) can be represented by a Fourier series

$$u_1 = \sum_n [G_n(a,\psi)\cos n\varphi + H_n(a,\psi)\sin n\varphi] \quad (17)$$

From the condition that the function u_1 does not contain any resonance terms, we conclude that the function u_1 does not contain $\cos\varphi$, $\sin\varphi$.

Substituting Eqs. (15) and (17) into Eq. (11) leads to

$$\begin{aligned}&\frac{\Omega^2}{4}\sum_n (n^2-1)\left[\left(\Omega\frac{n}{2}G_n - \alpha H_n\right)\sin n\varphi - \left(\Omega\frac{n}{2}H_n + \alpha G_n\right)\cos n\varphi\right] \\ &-\left(\frac{\Omega^2}{2}A_1 + \Omega\alpha aB_1\right)\cos\varphi + \left(\frac{\Omega^2}{2}aB_1 - \Omega\alpha A_1\right)\sin\varphi \\ &= ac\cos\varphi\cos\Omega t - \sum_{m=0}^{\infty}(r_m\cos m\varphi + s_m\sin m\varphi)\end{aligned} \quad (18)$$

On the other hand, we have

$$\begin{aligned}\cos\varphi\cos\Omega t &= \cos\varphi\cos(2\varphi - 2\psi) \\ &= \frac{1}{2}[\cos(-\varphi+2\psi) + \cos(3\varphi - 2\psi)] \\ &= \frac{1}{2}(\cos 2\psi\cos\varphi + \sin 2\psi\sin\varphi + \cos 2\psi\cos 3\varphi \\ &\quad + \sin 2\psi\sin 3\varphi)\end{aligned} \quad (19)$$

Substituting Eq. (19) into Eq. (18), we have

$$\frac{\Omega^2}{4}\sum_n (n^2-1)\left[\left(\Omega\frac{n}{2}G_n-\alpha H_n\right)\sin n\varphi-\left(\Omega\frac{n}{2}H_n+\alpha G_n\right)\cos n\varphi\right]$$
$$-\left(\frac{\Omega^2}{2}A_1+\Omega\alpha aB_1\right)\cos\varphi+\left(\frac{\Omega^2}{2}aB_1-\Omega\alpha A_1\right)\sin\varphi$$
$$=\frac{ac}{2}(\cos 2\psi\cos\varphi+\sin 2\psi\sin\varphi+\cos 2\psi\cos 3\varphi+\sin 2\psi\sin 3\varphi)$$
$$-\sum_{m=0}^{\infty}(r_m\cos m\varphi+s_m\sin m\varphi) \tag{20}$$

Comparing the coefficients of the harmonic functions $\cos\varphi$, $\sin\varphi$ on both sides of Eq. (20), we lead to

$$\begin{cases}\dfrac{\Omega^2}{2}A_1+\Omega\alpha aB_1=-\dfrac{ac}{2}\cos 2\psi+r_1\\ \Omega\alpha A_1-\dfrac{\Omega^2}{2}aB_1=-\dfrac{ac}{2}\sin 2\psi+s_1\end{cases} \tag{21}$$

Comparing the coefficients of other harmonic functions, we have

$$\begin{cases}\dfrac{\Omega^2}{4}(n^2-1)\left(\Omega\dfrac{n}{2}H_n+\alpha G_n\right)=r_n-\dfrac{ac}{2}\cos 2\psi\delta_{3n}\\ \dfrac{\Omega^2}{4}(n^2-1)\left(\Omega\dfrac{n}{2}G_n-\alpha H_n\right)=-s_n+\dfrac{ac}{2}\sin 2\psi\delta_{3n}\end{cases} \tag{22}$$

Here, $(n\neq 1)$ and

$$\delta_{3n}=\begin{cases}0 & (n\neq 3)\\ 1 & (n=3)\end{cases} \tag{23}$$

Equations (21) and (22) yield

$$G_n=\frac{\alpha r_n-\dfrac{\Omega}{2}ns_n+\left(\dfrac{\Omega}{4}nac\sin 2\psi-\dfrac{\alpha}{2}ac\cos 2\psi\right)\delta_{3n}}{\dfrac{\Omega^2}{4}(n^2-1)\left(\alpha^2+\dfrac{\Omega^2}{4}n^2\right)}$$
$$H_n=\frac{\dfrac{\Omega}{2}nr_n+\alpha s_n-\left(\dfrac{\Omega}{4}nac\cos 2\psi+\dfrac{\alpha}{2}ac\sin 2\psi\right)\delta_{3n}}{\dfrac{\Omega^2}{4}(n^2-1)\left(\alpha^2+\dfrac{\Omega^2}{4}n^2\right)} \tag{24}$$

and

$$A_1=\frac{\alpha\langle f_0\sin\varphi\rangle+\omega\langle f_0\cos\varphi\rangle-\dfrac{\omega}{4}ac\cos 2\psi-\dfrac{1}{4}ac\alpha\sin 2\psi}{\omega(\alpha^2+\omega^2)}$$
$$B_1=\frac{\alpha\langle f_0\cos\varphi\rangle-\omega\langle f_0\sin\varphi\rangle+\dfrac{\omega}{4}ac\sin 2\psi-\dfrac{1}{4}ac\alpha\cos 2\psi}{\Omega a(\alpha^2+\omega^2)} \tag{25}$$

For simplicity of writing, the following symbol is used:

$$R_0=h_2\frac{\Omega^2}{4}a^2\sin^2\varphi\,\mathrm{sign}\left(-\frac{\Omega}{2}a\sin\varphi\right)$$
$$+a\delta_p\left(\frac{\Omega}{2}\right)^p\left[\cos\frac{p\pi}{2}\cos\varphi-\sin\frac{p\pi}{2}\sin\varphi\right] \tag{26}$$

where

$$\mathrm{sign}\,\dot{x}=\begin{cases}1 & (\dot{x}>0)\\ -1 & (\dot{x}<0)\\ 0 & (\dot{x}=0)\end{cases} \tag{27}$$

Equation (14) yields

$$\langle f_0\cos\varphi\rangle=-\sigma\frac{\Omega}{2}a\langle\sin\varphi\cos\varphi\rangle+\alpha\sigma a\langle\cos^2\varphi\rangle+ka^3\langle\cos^4\varphi\rangle$$
$$-h\frac{\Omega^3}{8}a^3\langle\sin^3\varphi\cos\varphi\rangle+\langle R_0\cos\varphi\rangle$$
$$=\frac{1}{2}\alpha\sigma a+\frac{3}{8}ka^3+\langle R_0\cos\varphi\rangle$$
$$\langle f_0\sin\varphi\rangle=-\sigma\frac{\Omega}{2}a\langle\sin^2\varphi\rangle+\alpha\sigma a\langle\cos\varphi\sin\varphi\rangle+ka^3\langle\cos^3\varphi\sin\varphi\rangle$$
$$-h\frac{\Omega^3}{8}a^3\langle\sin^4\varphi\rangle+\langle R_0\sin\varphi\rangle$$
$$=-\frac{1}{4}\sigma\Omega a-\frac{3}{64}h\Omega^3a^3+\langle R_0\sin\varphi\rangle \tag{28}$$

Substituting Eq. (28) into Eq. (25), we obtain in the first approximation the following averaged equations of Eq. (6):

$$\frac{da}{dt}=\frac{\varepsilon}{\alpha^2+\omega^2}\left[\frac{3}{8}(k-\alpha\omega^2h)a^3-\frac{1}{4}ac\cos 2\psi-\frac{ac}{2\Omega}\alpha\sin 2\psi+R_1\right]$$
$$\frac{d\psi}{dt}=\frac{\varepsilon}{2a(\alpha^2+\omega^2)}\left[\frac{1}{\Omega}(\alpha^2+\omega^2)\sigma a+\frac{3}{4\Omega}(\alpha k+\omega^4h)a^3\right.$$
$$\left.+\frac{ac}{4}\sin 2\psi-\frac{ac}{2\Omega}\alpha\cos 2\psi+R_2\right] \tag{29}$$

where

$$R_1=\langle R_0\cos\varphi\rangle+\frac{2\alpha}{\Omega}\langle R_0\sin\varphi\rangle,\quad R_2=\frac{2\alpha}{\Omega}\langle R_0\cos\varphi\rangle-\langle R_0\sin\varphi\rangle \tag{30}$$

From Eq. (26), one can obtain the averaging values $\langle R_0\cos\varphi\rangle$, $\langle R_0\sin\varphi\rangle$

$$\langle R_0\cos\varphi\rangle=h_2\left\langle\frac{\Omega^2}{4}a^2\,\mathrm{sing}\left(-\frac{\Omega}{2}a\sin\varphi\right)\sin^2\varphi\cos\varphi\right\rangle$$
$$+a\delta_p\left(\frac{\Omega}{2}\right)^p\left[\cos\frac{p\pi}{2}\langle\cos^2\varphi\rangle-\sin\frac{p\pi}{2}\langle\sin\varphi\cos\varphi\rangle\right]$$
$$=\frac{1}{2}a\delta_p\left(\frac{\Omega}{2}\right)^p\cos\frac{p\pi}{2} \tag{31}$$

$$\langle R_0\sin\varphi\rangle=h_2\left\langle\frac{\Omega^2}{4}a^2\,\mathrm{sign}\left(-\frac{\Omega}{2}a\sin\varphi\right)\sin^3\varphi\right\rangle$$
$$+a\delta_p\left(\frac{\Omega}{2}\right)^p\left[\cos\frac{p\pi}{2}\langle\cos\varphi\sin\varphi\rangle-\sin\frac{p\pi}{2}\langle\sin^2\varphi\rangle\right]$$
$$=-\frac{1}{3\pi}h_2\Omega^2a^2-\frac{1}{2}a\delta_p\left(\frac{\Omega}{2}\right)^p\sin\frac{p\pi}{2} \tag{32}$$

Substituting Eqs. (31) and (32) into Eq. (30) yields

$$R_1=\frac{1}{2}a\delta_p\omega^{p-1}\left(\omega\cos\frac{p\pi}{2}-\alpha\sin\frac{p\pi}{2}\right)-\frac{4\alpha}{3\pi}h_2\omega a^2 \tag{33}$$

$$R_2=\frac{1}{2}a\delta_p\omega^{p-1}\left(\alpha\cos\frac{p\pi}{2}+\omega\sin\frac{p\pi}{2}\right)+\frac{4}{3\pi}h_2\omega^2a^2 \tag{34}$$

Then Eq. (29) have the following forms:

$$\frac{da}{dt}=\frac{\varepsilon}{\alpha^2+\omega^2}\left[\frac{3}{8}(k-\alpha\omega^2 h)a^3-\frac{1}{4}ac\cos 2\psi-\frac{ac}{2\Omega}\alpha\sin 2\psi+\frac{1}{2}a\delta_p\omega^{p-1}\left(\omega\cos\frac{p\pi}{2}-\alpha\sin\frac{p\pi}{2}\right)-\frac{4\alpha}{3\pi}h_2\omega a^2\right]$$
$$\frac{d\psi}{dt}=\frac{\varepsilon}{2a(\alpha^2+\omega^2)}\left[\frac{1}{\Omega}(\alpha^2+\omega^2)\sigma a+\frac{3}{4\Omega}(\alpha k+\omega^4 h)a^3+\frac{ac}{4}\sin 2\psi--\frac{ac}{2\Omega}\alpha\cos 2\psi+\frac{1}{2}a\delta_p\omega^{p-1}\left(\alpha\cos\frac{p\pi}{2}+\omega\sin\frac{p\pi}{2}\right)+\frac{4}{3\pi}h_2\omega^2 a^2\right] \tag{35}$$

Thus, in the first approximation, we have a partial solution of the Eq. (1) in the form

$$x=a\cos\left(\frac{\Omega}{2}t+\psi\right) \tag{36}$$

where a,ψ are solutions of the Eq. (35).

3 The Parametric Resonance Oscillation of System in the First Order Approximation

3.1 Amplitude–Frequency Curve. The amplitude and the phase of the stationary oscillation of the system (29) are determined by the following relation:

$$\begin{cases}\frac{c\alpha}{2\Omega}a_0\sin 2\psi_0+\frac{c}{4}a_0\cos 2\psi_0=\frac{3}{8}(k-\alpha\omega^2 h)a_0^3+R_1\\ \frac{c\alpha}{2\Omega}a_0\cos 2\psi_0-\frac{c}{4}a_0\sin 2\psi_0=\frac{1}{\Omega}(\alpha^2+\omega^2)\sigma a_0+\frac{3}{4\Omega}(\alpha k+\omega^4 h)a_0^3+R_2\end{cases} \tag{37}$$

The elimination of ψ_0 in Eq. (37) yields the amplitude–frequency equation

$$\mathrm{W}(a_0,\Omega)=\left[\alpha\sigma+\frac{3}{4}ka_0^2+\frac{2\omega^2}{a_0(\alpha^2+\omega^2)}\left(R_1+\frac{2\alpha}{\Omega}R_2\right)\right]^2+\omega^2\left[\sigma+\frac{3}{4}\omega^2 ha_0^2+\frac{1}{a_0(\alpha^2+\omega^2)}(\Omega R_2-2\alpha R_1)\right]^2-\frac{c^2}{4}=0 \tag{38}$$

Substituting R_1 and R_2 from Eqs. (33) and (34) into Eq. (38), we obtain

$$\left(\alpha\sigma+\frac{3}{4}ka_0^2+\delta_p\omega^p\cos\frac{p\pi}{2}\right)^2+\omega^2\left(\sigma+\frac{3}{4}\omega^2 ha_0^2+\delta_p\omega^{p-1}\sin\frac{p\pi}{2}+\frac{8\omega}{3\pi}h_2 a_0\right)^2-\frac{c^2}{4}=0 \tag{39}$$

$$\Longleftrightarrow 81\pi^2(k^2+\omega^6 h^2)a_0^4+576\omega^5 h\pi h_2 a_0^3+\left(216\pi^2\alpha\sigma k+216\pi^2\delta_p k\omega^p\cos\frac{p\pi}{2}++216\pi^2\omega^4\sigma h+216\pi^2\omega^{p+3}h\delta_p\sin\frac{p\pi}{2}+1024\omega^4 h_2^2\right)a_0^2+768\pi h_2\omega^3\left(\sigma+\omega^{p-1}\delta_p\sin\frac{p\pi}{2}\right)a_0+144\pi^2\sigma^2(\alpha^2+\omega^2)+144\pi^2\omega^{2p}\delta_p^2+\quad+288\pi^2\sigma\delta_p\omega^p\left(\alpha\cos\frac{p\pi}{2}+\omega\sin\frac{p\pi}{2}\right)-36c^2\pi^2=0 \tag{40}$$

3.2 The Existential Conditions for Periodic Oscillation. It is supposed that the characteristic equation

$$\lambda^3+c_1\lambda^2+c_2\lambda+c_3=0 \tag{41}$$

has a pair of either complex roots or imaginary roots. We have the following definitions:

(a) If the characteristic equation Eq. (41) has a negative root and a pair of complex roots with a negative real pair: $\lambda_1=-\xi,\ \lambda_2=-\eta+i\Omega,\ \lambda_3=-\eta-i\Omega$, then we have the noncritical case.
(b) If the characteristic equation Eq. (41) has a negative root and a pair of imaginary roots: $\lambda_1=-\xi,\ \lambda_2=i\Omega,\ \lambda_3=i\Omega$, then we have the critical case. In addition, if $\Omega=p/q\omega$, where and q are integers and Ω is the exciting frequency, then we have the critical resonant case.

The noncritical case has been investigated in many books, and the critical case has been investigated in Refs. [5–7].

3.3 Stability Condition of the Approximate Solution. Let δa and $\delta\psi$ be the small perturbations and set $a=a_0+\delta a,\psi=\psi_0+\delta\psi$, where a_0 and ψ_0 are the stationary values of a and ψ, determined from Eq. (29). Inserting the above expressions into Eq. (37), the following variation equations are obtained:

$$\frac{d\delta a}{dt}=\frac{\varepsilon}{\alpha^2+\omega^2}\left\{\left[\frac{3}{4}(k-\alpha\omega^2 h)a_0^2+a_0\left(\frac{R_1}{a_0}\right)'\right]\delta a-\left[\frac{2}{\Omega}(\alpha^2+\omega^2)\sigma a_0+\frac{3}{2\Omega}(\alpha k+\omega^4 h)a_0^3+2R_2\right]\delta\psi\right\} \tag{42}$$

$$\frac{d\delta\psi}{dt}=\frac{\varepsilon}{2(\alpha^2+\omega^2)}\left\{\left[\frac{3}{2\Omega}(\alpha k+\omega^4 h)a^0+\left(\frac{R_2}{a_0}\right)'\right]\delta a+\left[\frac{3}{4}(k-\alpha\omega^2 h)a_0^2+\frac{2}{a_0}R_1\right]\delta\psi\right\} \tag{43}$$

The characteristic equation of the system according to Eqs. (42) and (43) is

$$\lambda^2-Z\lambda+S=0 \tag{44}$$

where notations Z and S are defined as

$$Z = \frac{\varepsilon}{\alpha^2+\omega^2}\left[\frac{3}{2}(k-\alpha\omega^2 h)a_0^2 + \frac{1}{a_0}(a_0 R_1)'\right] \tag{45}$$

$$S = \frac{\varepsilon^2 a_0}{4\omega^2(\alpha^2+\omega^2)}\frac{\partial W}{\partial a_0} \tag{46}$$

with W is defined by Eq. (38).

According to the stability criterion of Routh-Hurwitz [31], it is obtained

$$3(k-\alpha\omega^2 h)a_0^3 + 2(a_0 R_1)' < 0 \tag{47}$$

$$\frac{\partial W}{\partial a_0} > 0 \tag{48}$$

From Eqs. (33), (34), (47), and (48), the stability condition of stationary oscillation is determined by the following relations:

$$3(k-\alpha\omega^2 h)a_0^3 + 2a_0\delta_p\omega^{p-1}\left(\omega\cos\frac{p\pi}{2} - \alpha\sin\frac{p\pi}{2}\right) - \frac{8\alpha}{\pi}h_2\omega a^2 < 0 \tag{49}$$

$$\begin{aligned} &3ka_0\left(\alpha\sigma + \frac{3}{4}ka_0^2 + \delta_p\omega^p\cos\frac{p\pi}{2}\right) \\ &+2\omega^2\left(\sigma + \frac{3}{4}\omega^2 h a_0^2 + \delta_p\omega^{p-1}\sin\frac{p\pi}{2} + \frac{8\omega}{3\pi}h_2 a_0\right) \\ &\left(\frac{3}{2}\omega^2 h a^0 + \frac{8\omega}{3\pi}h_2\right) > 0 \end{aligned} \tag{50}$$

4 Simulation Results

In order to demonstrate the effects of fractional-order damper on the harmonic vibration of a third-order nonlinear parametric system, a set of basic parameters are selected as follows:

$$\omega = 1, \quad \alpha = 1, \quad \varepsilon = 1, \quad \delta_p = 0.01, \quad p = 0.5, \quad k = -0.1,$$

$$h = 0.01, \quad h_2 = 0.01, \quad c = 0.05, \quad \eta = \frac{\Omega}{2\omega}$$

The data used in this section are selected according to our research experience, not from experimental models. The differential equation of oscillation of the system (1) is reduced to form

$$\begin{aligned} &\dddot{x} + \ddot{x} + \dot{x} + x + 1[-0.1x^3 + 0.01\dot{x}^3 + 0.001\dot{x}^2\ \mathrm{sign}\,\dot{x} \\ &+ 0.01D^{1/2}x - 0.05x\cos\Omega t] = 0 \end{aligned} \tag{51}$$

Using the amplitude–frequency equation according to Eq. (51), the amplitude–frequency curves can be plotted as in Figs. 1–7, where the solid line is for stable solution, the dashed line is for unstable one.

When the coefficient δ_p changes, the fractional order $p = 0.5$, the frequency–amplitude curves are shown in Fig. 1. If the coefficient $\delta_p = 0.01$ and the fractional order p change, the frequency–amplitude curves are shown in Fig. 2. We found that when the fractional order p increases, the amplitude of the oscillation decreases, when the coefficient of the fractional derivative increases, the amplitude of the oscillation does not increase, but the phase of oscillation changes.

Figure 3 shows the amplitude–frequency curves corresponding to different values of the friction coefficient h_2 and $\delta_p = 0.01, p = 0.5$. It also shows the important influence of the coefficient of friction on the frequency–amplitude curve. When the coefficient of friction h_2 increases, the amplitude of the oscillation decreases.

With the parameter set selected above, when $\delta_p = 0$; $p = 0.5$; $h_2 = 0.01$, from Eqs. (39) and (2) we deduce the equation for the frequency–amplitude curve as

$$\left(1-\eta^2-\frac{0.3}{4}a_0^2\right)^2 + \left(1-\eta^2+\frac{0.03}{4}a_0^2+\frac{0.08}{3\pi}a_0\right)^2 - \frac{0.05^2}{4} = 0 \tag{52}$$

When $\delta_p = 0.01$; $p = 0.5$; $h_2 = 0.01$, the equation for the amplitude–frequency curve has the following form:

$$\begin{aligned} &\left(1-\eta^2-\frac{0.3}{4}a_0^2+0.005\sqrt{2}\right)^2 \\ &+\left(1-\eta^2+\frac{0.03}{4}a_0^2+\frac{0.08}{3\pi}a_0+0.005\sqrt{2}\right)^2 - \frac{0.05^2}{4} = 0 \end{aligned} \tag{53}$$

Using the symbol $\eta_1^2 = \eta^2 - 0.005\sqrt{2}$, Eq. (53) has the form as the Eq. (52). Therefore, if we translate the frequency–amplitude curve when $\delta_p = 0$ to the right one segment $0.005\sqrt{2}$, we will obtain the graph of the amplitude–frequency curve corresponding to $\delta_p = 0.01$. We can easily see these properties in Figs. 4–6.

The graphs in Figs. 4–6 show the dependence of the amplitude–frequency curve on the parameters δ_p, p, and h_2. The crossed regions in Figs. 4–6 are unstable regions where

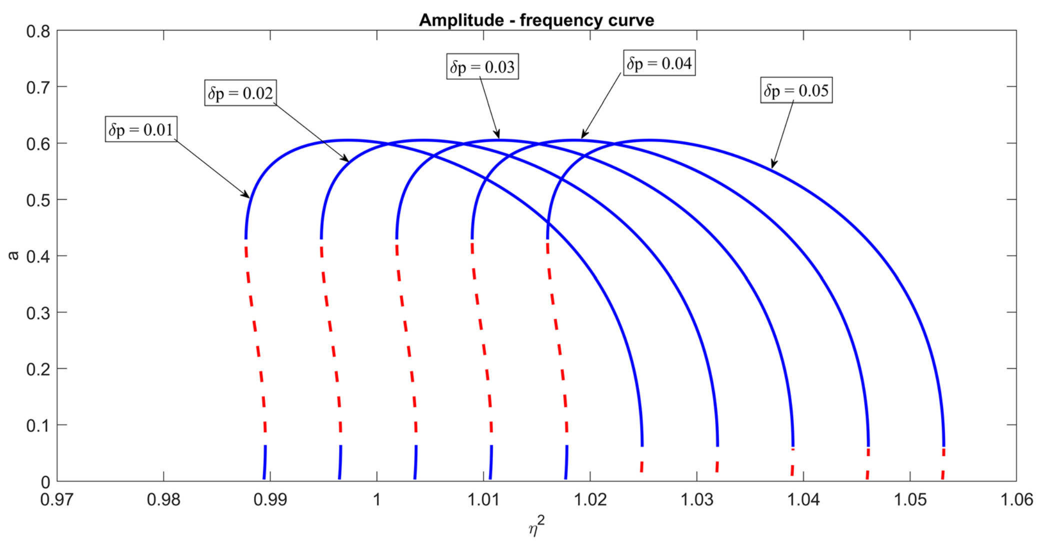

Fig. 1 The amplitude–frequency curve, where $p = 0.5$, and different values of δ_p

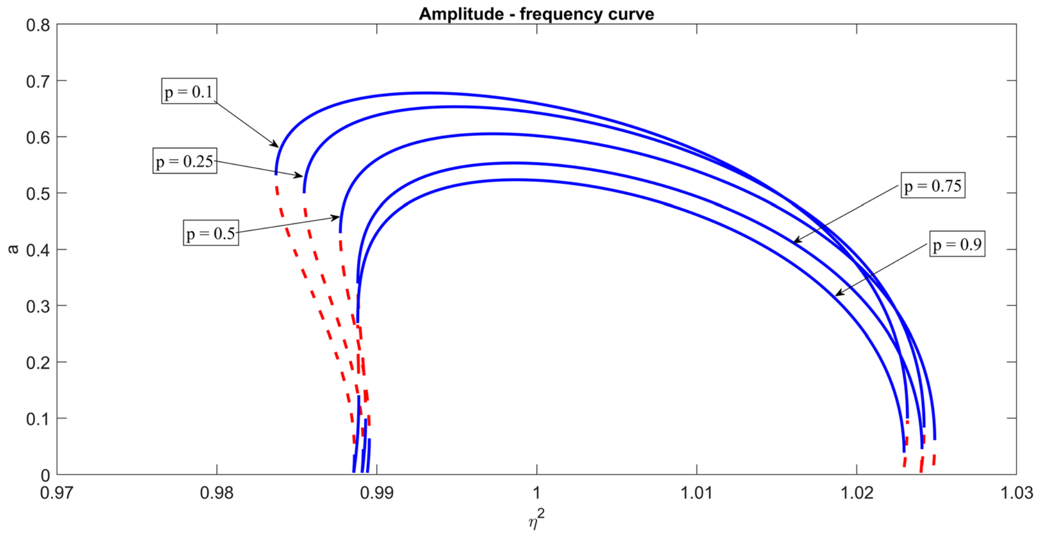

Fig. 2 The amplitude–frequency curve, where $\delta_p = 0.01$ and p changes

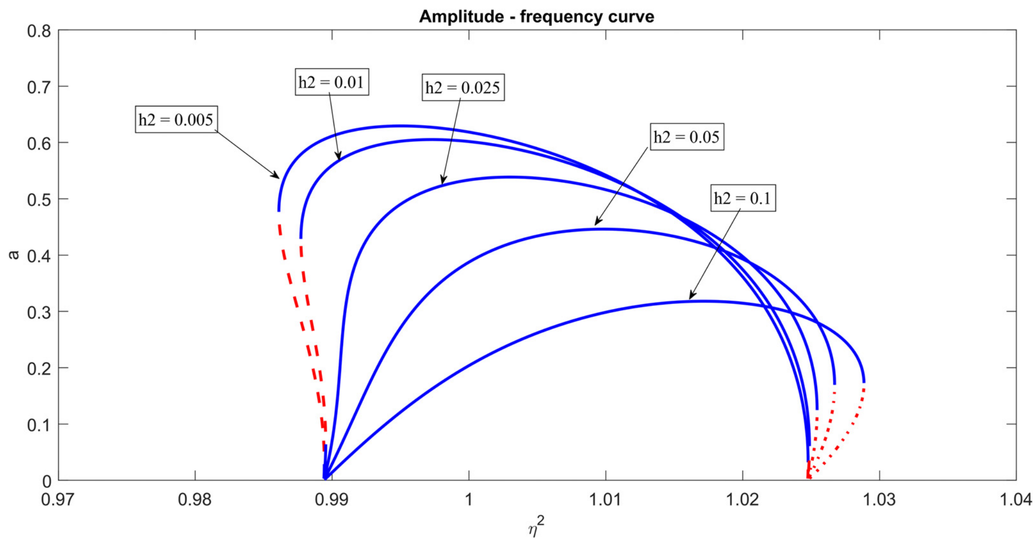

Fig. 3 The amplitude–frequency curve, where $\delta_p = 0.01$, $p = 0.5$, and h_2 changes

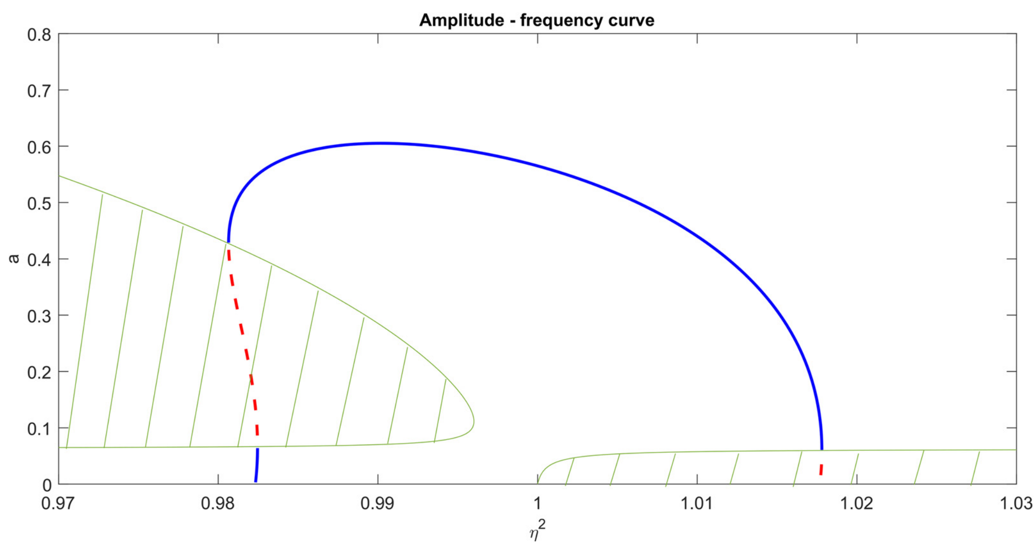

Fig. 4 The amplitude–frequency curve, where $\delta_p = 0$; $p = 0.5$; $h_2 = 0.01$

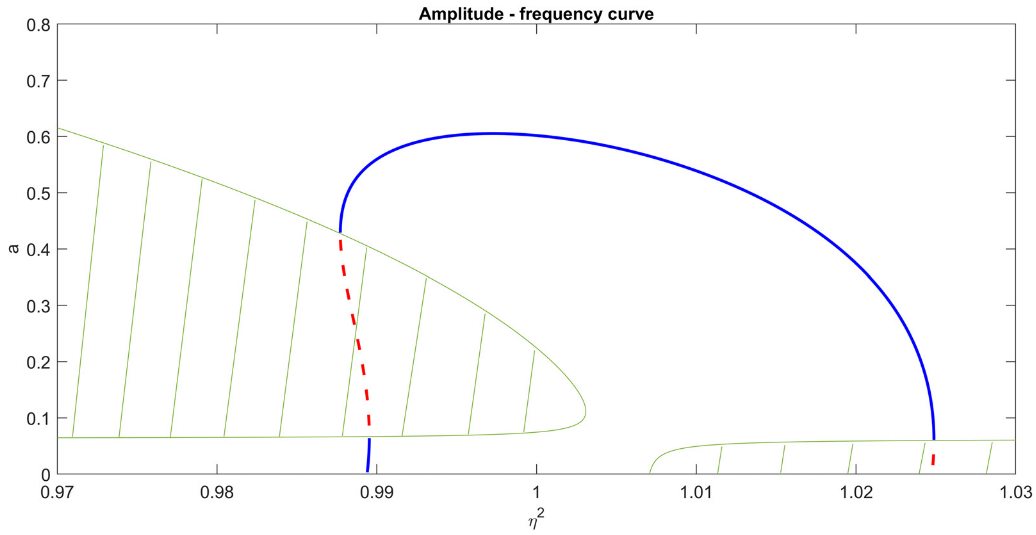

Fig. 5 The amplitude–frequency curve, where $\delta_p = 0.01$; $p = 0.5$; $h_2 = 0.01$

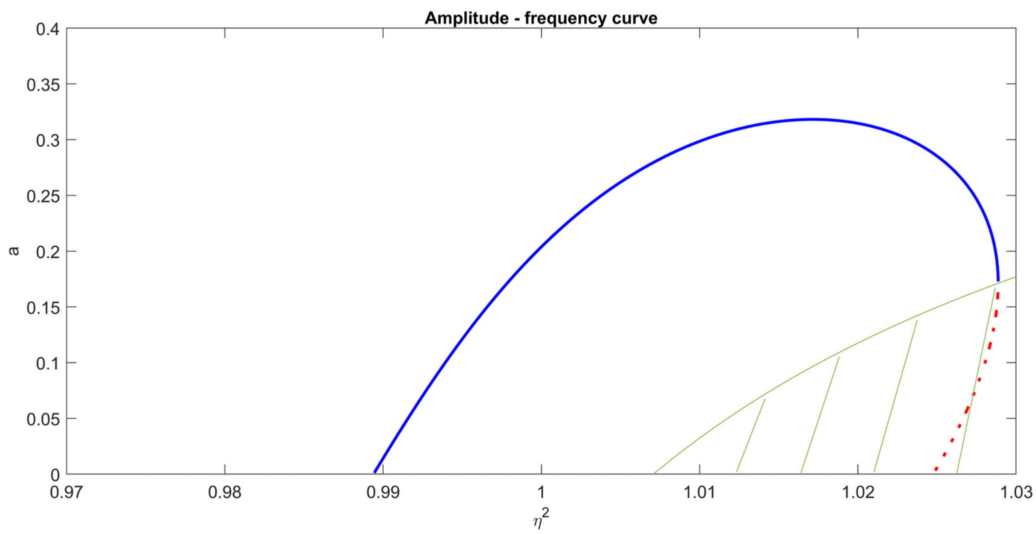

Fig. 6 The amplitude–frequency curve, where $\delta_p = 0.01$; $p = 0.5$; $h_2 = 0.1$

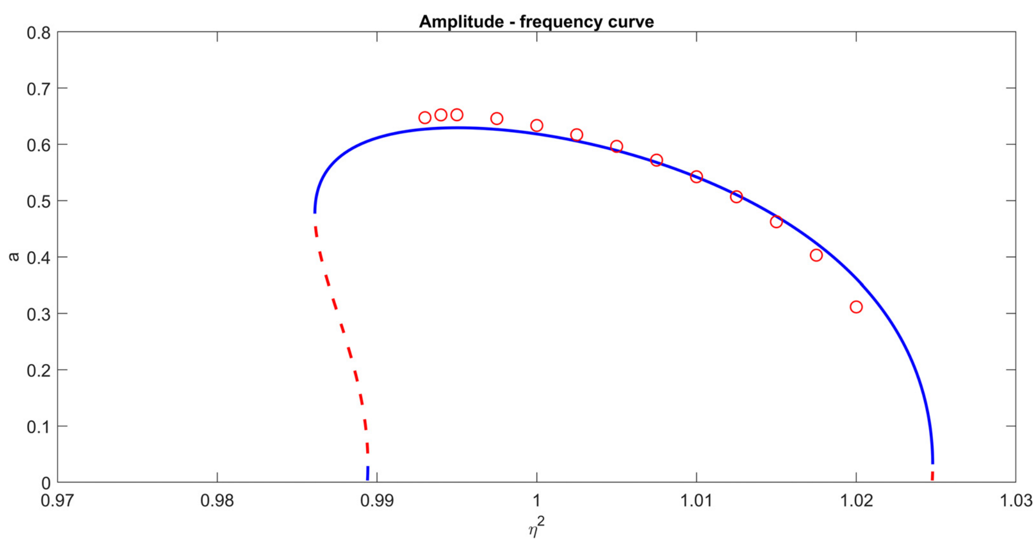

Fig. 7 The amplitude–frequency curve, where $\delta_p = 0.01$; $p = 0.5$; $h_2 = 0.005$

inequalities (49) and (50) are not satisfied. It is noted that the unstable area shifts from left to right as h changes from 0.01 to 0.1.

It should be noted that in Fig. 7 the solid curve is the solution calculated by analytical method, the circles denote the solution calculated by numerical integration [35]. Figure 7 clearly shows that there is good agreement between the numerical and analytical results.

5 Conclusions

In this paper, the parametric resonance of a third-order nonlinear vibration system with dynamic friction and fractional damping was investigated by the asymptotic method. The new findings made in this study are summarized as follows:

(1) Using the asymptotic method, an approximate solution expression of the third-order nonlinear system with dynamic friction and fractional damping has been built.
(2) The amplitude and the phase of the stationary oscillation of the approximate solution are determined. The stability conditions of the approximate solution have been studied.
(3) The effects of the fractional coefficient, the fractional-order and the dynamic coefficient on the approximate solution were characterized by the equivalent damping coefficient and the equivalent stiffness coefficient, which were also illustrated by some typical amplitude-frequency curves.

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Appendix

A formula for calculating the fractional-order derivative of the trigonometric functions $a(t)\cos[\Omega t+\psi(t)]$ and $a(t)\sin[\Omega t+\psi(t)]$

Theorem. *It is assumed that the frequency Ω is a constant, and a(t) and $\psi(t)$ are given by the differential equations*

$$\begin{aligned}\frac{da(t)}{dt} &= \varepsilon A_1(a,\psi) + \varepsilon^2 A_2(a,\psi) + \cdots \\ \frac{d\psi(t)}{dt} &= \varepsilon B_1(a,\psi) + \varepsilon^2 B_2(a,\psi) + \cdots\end{aligned} \tag{A1}$$

where ε is a small positive parameter, we have the following approximate expressions:

$$\begin{aligned}D_t^p[a(t)\cos(\Omega t+\psi(t))] &= \Omega^p a(t)\cos\left[\Omega t+\psi(t)+p\frac{\pi}{2}\right] + \varepsilon[\cdot] \\ D_t^p[a(t)\sin(\Omega t+\psi(t))] &= \Omega^p a(t)\sin\left[\Omega t+\psi(t)+p\frac{\pi}{2}\right] + \varepsilon[\cdot]\end{aligned} \tag{A2}$$

Proof. Using the Leibniz's rule for fractional differentiation of a product of two functions f and g [1], it is obtained

$$\begin{aligned}D^p[f(t)g(t)] &= \sum_{k=0}^{\infty}\binom{p}{k}D^k f(t) D^{p-k}g(t) \\ &= f(t)D^p g(t) + \sum_{k=1}^{\infty}\binom{p}{k}D^k f(t)D^{p-k}g(t)\end{aligned} \tag{A3}$$

By the choice $f=a\ (t)$, $g=\cos(\Omega t+\psi)=\cos\varphi(t)$, according to Eq. (A3) we have

$$\begin{aligned}D^p[a\cos(\Omega t+\psi)] &= a(t)D^p[\cos(\Omega t+\psi)] \\ &\quad + \sum_{k=1}^{\infty}\binom{p}{k}D^k[a(t)]D^{p-k}[\cos(\Omega t+\psi)] \\ &\approx a(t)D^p[\cos(\Omega t+\psi)] \\ &\quad + \binom{p}{1}D^1[a(t)]D^{p-1}[\cos(\Omega t+\psi)] \\ &\quad + \binom{p}{2}D^2[a(t)]D^{p-2}[\cos(\Omega t+\psi)] + \cdots\end{aligned} \tag{A4}$$

Now, we compute the fractional derivatives $D^\alpha[\cos(\Omega t+\psi(t))]$ with $\alpha = p,\ p-1,\ p-2$. Using the linearity of the fractional differential operator, we have

$$\begin{aligned}D^\alpha[\cos(\Omega t+\psi(t))] &= D^\alpha[\cos\Omega t\cos\psi - \sin\Omega t\sin\psi] \\ &= D^\alpha[\cos\Omega t\cos\psi] - D^\alpha[\sin\Omega t\sin\psi]\end{aligned} \tag{A5}$$

For $f=\cos\psi(t),\ g=\cos\Omega t$, according to Eq. (A3) we have

$$D^\alpha[\cos\psi\cos\Omega t] = \cos\psi D^\alpha(\cos\Omega t) + \sum_{k=1}^{\infty}\binom{\alpha}{k}D^k(\cos\psi)D^{\alpha-k}(\cos\Omega t) \tag{A6}$$

By similar calculations, we obtain

$$D^\alpha[\sin\psi\cos\Omega t] = \sin\psi D^\alpha(\sin\Omega t) + \sum_{k=1}^{\infty}\binom{\alpha}{k}D^k(\sin\psi)D^{\alpha-k}(\sin\Omega t) \tag{A7}$$

By substituting Eqs. (A6), (A7) into Eq. (A5), it is obtained

$$\begin{aligned}D^\alpha[\cos(\Omega t+\psi)] &= \cos\psi D^\alpha(\cos\Omega t) - \sin\psi D^\alpha(\sin\Omega t) \\ &\quad + \sum_{k=1}^{\infty}\binom{\alpha}{k}D^k(\cos\psi)D^{\alpha-k}(\cos\Omega t) \\ &\quad - \sum_{k=1}^{\infty}\binom{\alpha}{k}D^k(\sin\psi)D^{\alpha-k}(\sin\Omega t)\end{aligned} \tag{A8}$$

By substituting Eq. (A8) into Eq. (A4) yields

$$\begin{aligned}D^p[a\cos(\Omega t+\psi)] &= a(t)\{\cos\psi D^p[\cos(\Omega t)] - \sin\psi D^p[\sin(\Omega t)] \\ &\quad + \sum_{k=1}^{\infty}\binom{p}{k}D^k[\cos\psi]D^{p-k}[\cos(\Omega t)] \\ &\quad - \sum_{k=1}^{\infty}\binom{p}{k}D^k[\sin\psi]D^{p-k}[\sin(\Omega t)] \\ &\quad + \binom{p}{1}D^1[a(t)]\{\cos\psi D^{p-1}[\cos(\Omega t)] \\ &\quad - \sin\psi D^{p-1}[\sin(\Omega t)] \\ &\quad + \sum_{k=1}^{\infty}\binom{p-1}{k}D^k[\cos\psi]D^{p-1-k}[\cos(\Omega t)] \\ &\quad - \sum_{k=1}^{\infty}\binom{p-1}{k}D^k[\sin\psi]D^{p-1-k}[\sin(\Omega t)]\} \\ &\quad + \binom{p}{2}D^2[a(t)]\{\cos\psi D^{p-2}[\cos(\Omega t)] \\ &\quad - \sin\psi D^{p-2}[\sin(\Omega t)] \\ &\quad + \sum_{k=1}^{\infty}\binom{p-2}{k}D^k[\cos\psi]D^{p-2-k}[\cos(\Omega t)] \\ &\quad - \sum_{k=1}^{\infty}\binom{p-2}{k}D^k[\sin\psi]D^{p-2-k}[\sin(\Omega t)]\} + \cdots\end{aligned} \tag{A9}$$

Using the assumptions (A1), we have

$$
\begin{aligned}
D^1[a] &= \frac{da}{dt} = \varepsilon A_1 + \varepsilon^2 A_2 + \cdots \\
D^2[a] &= \frac{d^2 a}{dt^2} = \varepsilon \frac{dA_1}{dt} + \varepsilon^2 \frac{dA_2}{dt} + \cdots \\
D^1[\cos\psi] &= \frac{d(\cos\psi)}{dt} = -\sin\psi \frac{d\psi}{dt} = -\sin\psi\left(\varepsilon B_1 + \varepsilon^2 B_2 + \cdots\right) \\
D^2[\cos\psi] &= \frac{d^2(\cos\psi)}{dt^2} = -\frac{d\left(\sin\psi \frac{d\psi}{dt}\right)}{dt} = -\cos\psi\left(\frac{d\psi}{dt}\right)^2 \\
&\quad - \sin\psi \frac{d^2\psi}{dt^2} \\
&= -\cos\psi\left(\varepsilon B_1 + \varepsilon^2 B_2 + \cdots\right)^2 \\
&\quad - \sin\psi\left(\varepsilon \frac{dB_1}{dt} + \varepsilon^2 \frac{dB_2}{dt} + \cdots\right) \\
D^1[\sin\psi] &= \frac{d(\sin\psi)}{dt} = \cos\psi \frac{d\psi}{dt} = \cos\psi\left(\varepsilon B_1 + \varepsilon^2 B_2 + \cdots\right) \\
D^2[\sin\psi] &= \frac{d^2(\sin\psi)}{dt^2} = \frac{d\left(\cos\psi \frac{d\psi}{dt}\right)}{dt} = -\sin\psi\left(\frac{d\psi}{dt}\right)^2 \\
&\quad + \cos\psi \frac{d^2\psi}{dt^2} \\
&= -\sin\psi\left(\varepsilon B_1 + \varepsilon^2 B_2 + \cdots\right)^2 \\
&\quad + \cos\psi\left(\varepsilon \frac{dB_1}{dt} + \varepsilon^2 \frac{dB_2}{dt} + \cdots\right)
\end{aligned}
\tag{A10}
$$

Substituting Eqs. (A10) into Eq. (A9) it is obtained

$$
\begin{aligned}
D^p[a\cos(\Omega t + \psi)] = a(t)\{\cos\psi D^p[\cos(\Omega t)] - \sin\psi D^p[\sin(\Omega t)]\} \\
+ \varepsilon[.]
\end{aligned}
\tag{A11}
$$

Using the formulas [36]

$$
\begin{aligned}
D_x^p[\cos(\omega x)] &= \omega^p \cos\left(\omega x + p\frac{\pi}{2}\right), \\
D_x^p[\sin(\omega x)] &= \omega^p \sin\left(\omega x + p\frac{\pi}{2}\right)
\end{aligned}
\tag{A12}
$$

the formula (A11) can be transformed in the following form:

$$
\begin{aligned}
&D^p[a\cos(\Omega t + \psi)] \\
&= a(t)\left\{\cos\psi \Omega^p \cos\left(\Omega t + p\frac{\pi}{2}\right) - \sin\psi \Omega^p \sin\left(\Omega t + p\frac{\pi}{2}\right)\right\} + \varepsilon[.] \\
&= \Omega^p a(t)\cos\left(\Omega t + \psi + p\frac{\pi}{2}\right) + \varepsilon[.]
\end{aligned}
\tag{A13}
$$

or

$$
D_t^p[a(t)\cos(\Omega t + \psi(t))] = \Omega^p a(t)\cos\left[\Omega t + \psi(t) + p\frac{\pi}{2}\right] + \varepsilon[.]
\tag{A14}
$$

Similarly, we can prove the formula

$$
D_t^p[a(t)\sin(\Omega t + \psi(t))] = \Omega^p a(t)\sin\left[\Omega t + \psi(t) + p\frac{\pi}{2}\right] + \varepsilon[.]
\tag{A15}
$$

■

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