Dear Author,

Here are the proofs of your article.

- You can submit your corrections **online**, via **e-mail** or by **fax**.
- For **online** submission please insert your corrections in the online correction form. Always indicate the line number to which the correction refers.
- You can also insert your corrections in the proof PDF and **email** the annotated PDF.
- For fax submission, please ensure that your corrections are clearly legible. Use a fine black pen and write the correction in the margin, not too close to the edge of the page.
- Remember to note the **journal title**, **article number**, and **your name** when sending your response via e-mail or fax.
- **Check** the metadata sheet to make sure that the header information, especially author names and the corresponding affiliations are correctly shown.
- **Check** the questions that may have arisen during copy editing and insert your answers/ corrections.
- **Check** that the text is complete and that all figures, tables and their legends are included. Also check the accuracy of special characters, equations, and electronic supplementary material if applicable. If necessary refer to the *Edited manuscript*.
- The publication of inaccurate data such as dosages and units can have serious consequences. Please take particular care that all such details are correct.
- Please **do not** make changes that involve only matters of style. We have generally introduced forms that follow the journal's style.
  Substantial changes in content, e.g., new results, corrected values, title and authorship are not allowed without the approval of the responsible editor. In such a case, please contact the Editorial Office and return his/her consent together with the proof.
- If we do not receive your corrections **within 48 hours**, we will send you a reminder.
- Your article will be published **Online First** approximately one week after receipt of your corrected proofs. This is the **official first publication** citable with the DOI. **Further changes are, therefore, not possible.**
- The **printed version** will follow in a forthcoming issue.

Please note

After online publication, subscribers (personal/institutional) to this journal will have access to the complete article via the DOI using the URL: http://dx.doi.org/[DOI].
If you would like to know when your article has been published online, take advantage of our free alert service. For registration and further information go to: http://www.link.springer.com.

Due to the electronic nature of the procedure, the manuscript and the original figures will only be returned to you on special request. When you return your corrections, please inform us if you would like to have these documents returned.

Metadata of the article that will be visualized in OnlineFirst

ArticleTitle	New results on finite-time stability of fractional-order neural networks with time-varying delay	
Article Sub-Title		
Article CopyRight	The Author(s), under exclusive licence to Springer-Verlag London Ltd., part of Springer Nature (This will be the copyright line in the final PDF)	
Journal Name	Neural Computing and Applications	

Corresponding Author	Family Name	**Phat**
	Particle	
	Given Name	**Vu N.**
	Suffix	
	Division	
	Organization	ICRTM, Institute of Mathematics, VAST
	Address	18 Hoang Quoc Viet Road, Hanoi, 10307, Vietnam
	Phone	
	Fax	
	Email	vnphat@math.ac.vn
	URL	
	ORCID	
Author	Family Name	**Thanh**
	Particle	
	Given Name	**Nguyen T.**
	Suffix	
	Division	Department of Basic Science
	Organization	University of Mining and Geology
	Address	Hanoi, Vietnam
	Phone	
	Fax	
	Email	trthanh1999@gmail.com
	URL	
	ORCID	
Author	Family Name	**Niamsup**
	Particle	
	Given Name	**P.**
	Suffix	
	Division	RCMAM, Department of Mathematics
	Organization	Chiang Mai University
	Address	Chiang Mai, 50200, Thailand
	Phone	
	Fax	
	Email	piyapong.n@cmu.ac.th
	URL	

ORCID

Schedule	Received	3 April 2021
	Revised	
	Accepted	18 July 2021

Abstract

In this paper, we propose an analytical approach based on the Laplace transform and Mittag–Leffler functions combining with linear matrix inequality techniques to study finite-time stability of fractional-order neural networks (FONNs) with time-varying delay. The concept of finite-time stability is extended to the fractional-order neural networks and the delay function is assumed to be non-differentiable, but continuous and bounded. We first prove some important lemmas on the existence of solutions and on estimation of the Caputo derivative of specific quadratic functions. Then, new delay-dependent sufficient conditions for finite-time stability of FONNs with time-varying delay are derived in terms of a tractable linear matrix inequality and Mittag–Leffler functions. Finally, a numerical example with simulations is provided to demonstrate the effectiveness and validity of the theoretical results.

Keywords (separated by '-'): Fractional derivative - Neural networks - Finite-time stability - Time-varying delay - Laplace transform - Mittag–Leffler functions

Footnote Information

Neural Computing and Applications
https://doi.org/10.1007/s00521-021-06339-2

ORIGINAL ARTICLE

New results on finite-time stability of fractional-order neural networks with time-varying delay

Nguyen T. Thanh[1] · P. Niamsup[2] · Vu N. Phat[3]

Received: 3 April 2021 / Accepted: 18 July 2021

Abstract
In this paper, we propose an analytical approach based on the Laplace transform and Mittag–Leffler functions combining with linear matrix inequality techniques to study finite-time stability of fractional-order neural networks (FONNs) with time-varying delay. The concept of finite-time stability is extended to the fractional-order neural networks and the delay function is assumed to be non-differentiable, but continuous and bounded. We first prove some important lemmas on the existence of solutions and on estimation of the Caputo derivative of specific quadratic functions. Then, new delay-dependent sufficient conditions for finite-time stability of FONNs with time-varying delay are derived in terms of a tractable linear matrix inequality and Mittag–Leffler functions. Finally, a numerical example with simulations is provided to demonstrate the effectiveness and validity of the theoretical results.

Keywords Fractional derivative · Neural networks · Finite-time stability · Time-varying delay · Laplace transform · Mittag–Leffler functions

1 Introduction

In the real world, neural networks have been found everywhere such as in weather forecasting and business processes because neural networks can create simulations and predictions for complex systems and relationships [1–6]. It is well-known that fractional-order systems (FOSs) have attracted much attention due to their important applications in various areas of applied sciences over the past decades [7–9]. Fractional analysis has been considered and developed in the context of neural networks as artificial neural networks, Hopfield neural networks, etc. In fact, the fractional-order derivative provides neurons with a fundamental and general computational ability that contributes to efficient information processing and frequency-independent phase shifts in oscillatory neuronal firings [10–16]. So far, most of the existing literature have concerned with Lyapunov asymptotic stability, however, in many practical cases, one concerns the system behavior on finite-time interval, i.e., finite-time stability (FTS) [17]. The concept of FTS has been developed to control problems, which concern the design of a admissible controllers ensuring the FTS of the closed-loop system. Many valuable results on finite-time control problems such as finite-time stabilization, finite-time optimal control, adaptive fuzzy finite-time optimal control, etc. have been obtained for this type of stability, see, [18–21] and the references therein. Therefore, problem of finite-time stability (FTS) for neural networks described by fractional differential equations has attracted a lot of attention from scientists. It is notable that most of the results on the stability of FOS neural networks did not consider time delay. In many practical applications, time delay is well-known to be unavoidable and it can cause oscillation or instability of the system.

✉ Vu N. Phat
vnphat@math.ac.vn

Nguyen T. Thanh
trthanh1999@gmail.com

P. Niamsup
piyapong.n@cmu.ac.th

[1] Department of Basic Science, University of Mining and Geology, Hanoi, Vietnam

[2] RCMAM, Department of Mathematics, Chiang Mai University, Chiang Mai 50200, Thailand

[3] ICRTM, Institute of Mathematics, VAST, 18 Hoang Quoc Viet Road, Hanoi 10307, Vietnam

There are various approaches to studying FTS for FOSs with delays including Lyapunov function method, Gronwall and Holder inequality approach, etc. The authors of [22–24] studied FTS of linear FOSs by using a generalized fractional Gronwall inequality lemma. In [25], Yang et al. studied FTS of fractional-order neural networks (FONNs) with delay. Chen et al. in [26] used some Holder-type inequalities to propose new criteria for FTS. Combining the Holder inequality and Gronwall inequality, Wu et al. [27] obtained sufficient conditions for FTS of FONNs with constant delay. Based on this approach the authors of [28, 29] developed the similar results for the systems with proportional constant delays. On the other hand, noting that the Lyapunov–Krasovskii function (LKF) method is one of the powerful techniques to studying stability of FOSs with delays, however, the LKF method can not be helpfully applied for fractional-order time-delay systems. The difficulty lies in the finding LKF to apply the fractional Lyapunov stability theorem. In [30–33], the authors used fractional Lyapunov stability theorem to find appropriate LKF for FOSs with time-varying delay, however, the proof of the main theorem provides a gap due to a wrong application of fractional Lyapunov stability theorem. Hence, it is worth investigating the stability of FONNs with time-varying delay. In the paper [34], the authors provided some sufficient conditions for the FTS of singular fractional-order systems with with time-varying delay. Very recently, to avoid finding LKF the authors of [35] employed fractional-order Razumikhin stability theorem to derive criteria for H_∞ control of FONNs with time-varying delay. To our knowledge, problem of FTS for fractional-order neural networks with time-varying delays has not yet been fully studied in the literature.

Motivated the above discussion, in this paper, we investigate problem of FTS for a class of FONNs with time-varying delay. Especially, the time-varying delay considered in the FONNs is only required to be continuous and interval bounded. The contribution of this paper is twofold. First, considering FONNs with interval time-varying delay, we propose some auxiliary lemmas on the existence of solutions and on estimating the Caputo derivative of some specific quadratic functions. Second, using a proposed analytical approach based on the factional calculus combining with LMI technique, we provide sufficient conditions for FTS. The conditions are established in terms of a tractable LMI and Mittag–Leffler functions. It should be noted that the proposed approach of Laplace transforms and inf–sup method has not yet seen in the field of FONNs with time-varying delay, and the stability conditions obtained in this paper are delay-dependent and novel.

The article is structured as follows. Section 2 presents formulation of the problem and some auxiliary technical lemmas. In Sect. 3, the main result on FTS is presented with an illustrative example and its simulation.

Notations. $\mathbb{R}^+$ denotes the set of all real positive numbers; $\mathbb{R}^n$ denotes the Euclidean $n-$ dimensional space with its scalar product $x^{top}y$; $\mathbb{R}^{n\times r}$ denotes the space of all $(n\times r)$-matrices; A^{top} denotes the transpose of A; matrix A is positive semi-definite $(A\geq 0)$ if $x^\top Ax\geq 0$, for all $x\in\mathbb{R}^n$; A is positive definite $(A>0)$ if $x^\top Ax>0$ for all $x\neq 0$; $A\geq B$ means $A-B\geq 0$; $C([-\tau,0],\mathbb{R}^n)$ denotes the set of vector valued continuous functions from $[-\tau,0]$ to $\mathbb{R}^n$;

2 Preliminaries

We first recall from [7] basic concepts of fractional calculus and some auxiliary results for the use in next section.

Definition 1 [7] For $\alpha\in(0,1)$ and $f\in L^1[0,T]$, the fractional integral $I^\alpha f(t)$, the Riemann derivative $D^\alpha_R f(t)$ and the Caputo derivative $D^\alpha_C \alpha f(t)$ of order α,, respectively, as

$$I^\alpha f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)ds,$$

$$D^\alpha_R f(t)=\frac{d}{dt}(I^{1-\alpha}f(t)),\quad D^\alpha_C f(t)=D^\alpha_R(f(t)-f(0)),$$

where $\Gamma(s)=\int\limits_0^\infty e^{-t}t^{s-1}dt, s>0,\ t\in[0,T]$ is the Gamma function.

The function

$$E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(n\alpha+\beta)}, z\in\mathbb{C},\ \alpha>0,\ \beta>0$$

denotes Mittag–Leffler function. The Laplace transform of the integrable function $g(.)$ is defined by $\mathcal{L}[g(t)](s)=\int\limits_0^\infty e^{-st}g(t)dt.$

Lemma 1 [7] *Assume that $f_1(.)$,$f_2(.)$ are exponentially bounded integrable functions on $\mathbb{R}^+$, and $0<\alpha<1,\beta>0$. Then*

(1) $\mathcal{L}[D^\alpha_C f_1(t)](s)=s^\alpha\mathcal{L}[f_1(t)](s)-s^{\alpha-1}f_1(0),$

(2) $\mathcal{L}[t^{\alpha-1}E_{\alpha,\alpha}(\beta t^\alpha)](s)=\dfrac{1}{s^\alpha-\beta},$

$$\mathcal{L}[E_\alpha(\beta t^\alpha)](s)=\frac{s^{\alpha-1}}{s^\alpha-\beta},$$

(3) $\mathcal{L}[f_1 * f_2(t)](s)=\mathcal{L}[f_1(t)](s)\cdot\mathcal{L}[f_2(t)](s),$

where $f_1(t)*f_2(t):=\int\limits_0^t f_1(t-\tau)f_2(\tau)d\tau.$

Consider a FONNs with time-varying delay:

Journal : Large 521 | Dispatch : 25-7-2021 | Pages : 8
Article No. : 6339 | LE | TYPESET
MS Code : NCAA-D-21-01271R1 | CP | DISK

$$\begin{cases} D_C^\alpha x_i(t) = -m_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_i(t)) + \sum_{j=1}^{n} b_{ij} g_j(x_j(t-d(t))), \\ x_i(\theta) = \phi_i(\theta), \theta \in [-d_2, 0], i = \overline{1,n}, \end{cases} \tag{1}$$

or in the matrix form:

$$D_C^\alpha x(t) = -Mx(t) + Ff(x(t)) + Gg(x(t-d(t))), \tag{2}$$

where $x(t) = (x_i(t), ..., x_n(t))^\top$ is the state; the delay $d(t)$ satisfies $0 < d_1 \le d(t) \le d_2,\ \forall t \ge 0$; $\phi(t) = (\phi_i(t), ..., \phi_n(t))^\top$ is the initial condition with the norm

$$\|\phi\| = \sup_{\theta \in [-d_2, 0]} \sqrt{\sum_{i=1}^{n} |\phi_i(\theta)|^2};$$

the variation functions

$$f(x) = (f_1(x_1)), \ldots, f_n(x_n))^\top,\ \ g(x) = (g_1(x_1)), \ldots, g_n(x_n))^\top,$$

satisfy $f(0) = 0$, $g(0) = 0$, and for all $\xi, \eta \in \mathbb{R}$, $i = \overline{1,n}$:

$$\begin{aligned} &\exists l_i > 0 : |f_i(\xi) - f_i(\eta)| \le l_i |\xi - \eta|, \\ &\exists k_i > 0 : |g_i(\xi) - g_i(\eta)| \le k_i |\xi - \eta|; \end{aligned} \tag{3}$$

$M = diag(m_1, m_2, \ldots, m_n)$; $F = (a_{ij})_{n\times n}$, $G = (b_{ij})_{n\times n}$ are the connections of the j^{th} neuron to the i^{th} neuron at time t.

Definition 2 Let c_1, c_2, T be given positive numbers. System (1) is FTS with respect to (c_1, c_2, T) if

$$\|\phi\|^2 \le c_1 \Rightarrow \|x(t)\|^2 \le c_2, \quad t \in [0, T].$$

Lemma 2 *If $\phi \in C([-d_2, 0], \mathbb{R}^n)$ and the condition (3) holds, then system (1) has a unique solution $x \in C([-d_2, T), \mathbb{R}^n)$.*

Proof From Volterra integral form of system (2) we have

$$x(t) = x(0) + I^\alpha[-Mx(t) + Ff(x(t)) + Gg(x(t-d(t)))],$$

and consider the function

$$H(y)(t) = \begin{cases} \phi(0) + I^\alpha[v_y(t)] & \text{if } t \ge 0, \\ \phi(t) & \text{if } t \in [-d_2, 0), \end{cases}$$

where

$$v_y(t) = -My(t) + Ff(y(t)) + Gg(y(t-d(t))).$$

Note that the function $v_y(t)$ is continuous on [0, T] if $y \in C([-d_2, T], \mathbb{R}^n)$. So we can see that function $H(\cdot)$ maps $C([-d_2, T], \mathbb{R}^n)$ into $C([-d_2, T], \mathbb{R}^n)$. In fact from the uniform continuity of $v_y(t)$ on [0, T], there is a $\delta > 0$ such that for all $t_1, t_2 \in [0, T]$, $t_2 \le t_1$, and

$$|t_1 - t_2| \le \delta \Rightarrow |v_y(t_1) - v_y(t_2)| \le \varepsilon,$$

hence

$$\begin{aligned} |H(y)(t_1) - H(y)(t_2)| &\le \frac{\varepsilon}{\Gamma(\alpha)} \\ &\Big|\int_0^{t_2} s^{\alpha-1} ds\Big| + \frac{1}{\Gamma(\alpha)} \sup_{s\in[0,T]} |v_y(s)| \Big|\int_{t_2}^{t_1} s^{\alpha-1} ds\Big| \\ &\le \frac{\varepsilon}{\Gamma(\alpha)} \frac{T^\alpha}{\alpha} + \frac{1}{\Gamma(\alpha)} \sup_{s\in[0,T]} |v(s)| \Big|\frac{t_2^\alpha}{\alpha} - \frac{t_1^\alpha}{\alpha}\Big|, \end{aligned}$$

which also shows the continuity of $H(y)(t)$ on $[-d_2, T]$. Next, for $t \in [0, T]$, $y, z \in C([-d_2, T], \mathbb{R}^n)$:

$$\begin{aligned} |v_y(t) - v_z(t)| &\le |M||y(t) - z(t)| + |F||f(y(t)) - f(z(t))| \\ &\quad + |G||g(y(t-d(t))) - g(z(t-d(t)))| \\ &\le (|M| + |F| \max_i l_i + |G| \max_i k_i) \\ &\quad \sup_{s\in[-d_2,T]} |y(s) - z(s)|, \end{aligned}$$

which leads to

$$|H(y)(t) - H(z)(t)| \le \frac{\gamma_1 t^\alpha}{\Gamma(\alpha)\alpha} \sup_{s\in[-d_2,T]} |y(s) - z(s)|,$$

where $\gamma_1 = |M| + |F| \max_i l_i + |G| \max_i k_i$. Similarly, by induction, we have for $m = 1, 2...$

$$|H^m(y)(t) - H^m(z)(t)| \le \gamma_1 \frac{t^{m\alpha}}{\Gamma(m\alpha+1)} \sup_{s\in[-d_2,T]} |y(s) - z(s)|,$$

$$\begin{aligned} &\sup_{s\in[-d_2,T]} |H^m(y)(s) - H^m(z)(s)| \\ &\le \frac{\gamma_1 T^{m\alpha}}{\Gamma(m\alpha+1)} \sup_{s\in[-d_2,T]} |y(s) - z(s)|. \end{aligned}$$

Besides, the space $C([-d_2, T], \mathbb{R}^n)$ with the norm $\|y\| = \sup_{s\in[-d_2,T]} |y(s)|$ is a Banach space. Hence,

$$H^m(\cdot) : C([-d_2, T], \mathbb{R}^n) \to C([-d_2, T], \mathbb{R}^n)$$

is a contraction map with this sup norm as m enough large. Applying the fixed-point theorem, we derive the existence of a unique solution $x \in C([-d_2, T], \mathbb{R}^n)$. □

Lemma 3 [34] *For $d > 0$ and $N > 0$, if function $S : [-d, N] \to \mathbb{R}^+$ is non-decreasing and satisfies*

$$S(t) \le aS(0) + bS(t-d), a > 1, b \ge 0, t \ge 0,$$

then

$$S(t) \le S(0)a \sum_{j=0}^{[N/d]+1} b^j,\ \forall t \in [0, N].$$

Journal : Large 521 Dispatch : 25-7-2021 Pages : 8
Article No. : 6339 LE TYPESET
MS Code : NCAA-D-21-01271R1 CP DISK

3 Main result

This section provides new conditions for FTS of system (1) in term of a tractable LMI and Mittag–Leffler condition. Before proving the theorem, let us denote $[d]$ by the integer part of d and

$$\gamma = \frac{d_2}{2\max\limits_i k_i^2},\ \mathbb{I}_n = diag\{1,\ldots,1\} \in \mathbb{R}^{n\times n},$$

$$E_{11} = -2PM - d_2P + \beta \max_i l_i^2 \mathbb{I}_n, E_{12} = PF,\ E_{21} = [PF]^\top,\ E_{13} = PG,$$

$$E_{31} = [PG]^\top,\ E_{22} = -\beta\mathbb{I}_n,\ E_{33} = -\gamma P,\ E_{44} = \mathbb{I}_n - P,$$

$$E_{55} = P - 2\mathbb{I}_n,\ \text{all the others } E_{ij} = 0.$$

Theorem 1 *Let c_1, c_2, T be given positive numbers. System (1) is FTS with respect to (c_1, c_2, T) if there exist a number $\beta > 0$ and a symmetric matrix $P > 0$ such that*

$$\begin{pmatrix} E_{11} & E_{12} & . & . & E_{15} \\ * & E_{22} & . & . & E_{25} \\ . & . & . & . & . \\ * & * & . & . & E_{55} \end{pmatrix} < 0 \tag{4}$$

$$\frac{\lambda_{max}(P)}{\lambda_{min}(P)} E_\alpha(d_2T^\alpha) \sum_{j=0}^{[T/d_1]+1} (E_\alpha(d_2T^\alpha) - 1)^j < \frac{c_2}{c_1}. \tag{5}$$

Proof Let us consider the following non-negative quadratic functional $V(x(t)) = x(t)^\top Px(t)$. Since the solution $x(t)$ may not be non-differentiable, we propose the following result on estimating Caputo derivative of $V(x(t))$. □

Lemma 4 *For the solution $x(t) \in C([-d_2, T], \mathbb{R}^n)$, the Caputo derivative $D_C^\alpha(V(x(t))) \in C([0,T], \mathbb{R}^n)$ exists and $D_C^\alpha[V(x(t))] \le 2x(t)^\top PD_C^\alpha x(t), \quad t \ge 0$.*

To prove the lemma, we note that $x(t) \in C([-d_2, T], \mathbb{R}^n)$ (by Lemma 2), the function

$$u(t) = -Mx(t) + Ff(x(t)) + Gg(x(t-d(t))),$$

is continuous on [0, T]. Hence, we get

$$\left|\frac{x(t)-x(0)}{t^\alpha} - \frac{u(0)}{\Gamma(\alpha+1)}\right| = \left|\frac{\int_0^t (t-s)^{\alpha-1}(u(s)-u(0))ds}{t^\alpha\Gamma(\alpha)}\right|$$
$$\le \sup_{s\in[0,t]}\left|u(s)-u(0)\right|\left|\frac{\int_0^t (t-s)^{\alpha-1}ds}{t^\alpha\Gamma(\alpha)}\right|$$
$$= \frac{1}{\Gamma(\alpha+1)}\sup_{s\in[0,t]}\left|u(s)-u(0)\right| \to 0,$$

as $t \to 0$. In the other words,

$$\gamma_0 := \lim_{t\to 0}\frac{x(t)-x(0)}{t^\alpha} = \frac{u(0)}{\Gamma(\alpha+1)}. \tag{6}$$

Consequently,

$$\lim_{t\to 0}\frac{V(x(t)) - V(x(0))}{t^\alpha} = 2\left(x(0), \frac{Pu(0)}{\Gamma(\alpha+1)}\right). \tag{7}$$

It is easy to calculate the following integral

$$\begin{aligned}\int_{\xi t}^t \frac{V(x(t)) - V(x(s))}{(t-s)^{\alpha+1}}ds &= \int_{\xi t}^t \frac{(x(t)-x(s), 2Px(t))}{(t-s)^{\alpha+1}}ds \\ &- \int_{\xi t}^t \frac{(x(t)-x(s), P[x(t)-x(s)])}{(t-s)^{\alpha+1}}ds \\ &= I_1(t,\xi) - I_2(t,\xi).\end{aligned} \tag{8}$$

From Theorem 2.2 of [36] it follows that $D_C^\alpha x = u \in C([0,T], \mathbb{R}^n)$, and when $\xi \to 1^-$, we have

$$\begin{aligned}|I_1(t,\xi)| &= \left|\left(\int_{\xi t}^t \frac{x(t)-x(\tau)}{(t-\tau)^{\alpha+1}}d\tau,\ 2Px(t)\right)\right| \\ &\le \sup_{0<t\le T}\left|\int_{\xi t}^t \frac{x(t)-x(\tau)}{(t-\tau)^{\alpha+1}}d\tau\right| 2\sup_{t\in[0,T]}|Px(t)| \to 0,\end{aligned} \tag{9}$$

when $\xi \to 1^-$, and

$$x = x(0) + \gamma_0 t^\alpha + x_0,\ x_0 \in H_0^\alpha[0,T],\ t \in (0,T].$$

Hence, for $0 \le \xi t \le \tau < t \le T,\ \xi \in (0,1]$, we obtain that

$$\begin{aligned}\left|\frac{x(t)-x(\tau)}{(t-\tau)^\alpha}\right| &\le \left|\gamma_0\frac{t^\alpha - s^\alpha}{(t-\tau)^\alpha}\right| + \left|\frac{x_0(t)-x_0(\tau)}{(t-\tau)^\alpha}\right| \\ &= \gamma_0\frac{(t-\tau)\alpha c^{\alpha-1}}{(t-\tau)^\alpha} + \left|\frac{x_0(t)-x_0(\tau)}{(t-\tau)^\alpha}\right|, \\ &\le k(\xi) := \gamma_0\alpha[1/\xi - 1]^{1-\alpha} \\ &+ \sup_{0\le\tau<t\le T, |t-\tau|\le T(1-\xi)}\left|\frac{x_0(t)-x_0(\tau)}{(t-\tau)^\alpha}\right|,\end{aligned}$$

where $c \in (\tau, t)$. Thus, as $\xi \to 1^-$, we get

$$\begin{aligned}|I_2(t,\xi)| &= \int_{\xi t}^t \frac{(x(t)-x(\tau), P[x(t)-x(\tau)])}{(t-\tau)^{\alpha+1}}d\tau \\ &\le \frac{T^\alpha(1-\xi)^\alpha}{\alpha}\|P\|k(\xi)^2 \to 0,\end{aligned} \tag{10}$$

because $k(\xi)$ is independent on τ, t, and $x_0 \in H_0^\alpha[0,T]$. From (8), (9), (10), as $\xi \to 1^-$,

Journal : Large 521 | Dispatch : 25-7-2021 | Pages : 8
Article No. : 6339 | □ LE | □ TYPESET
MS Code : NCAA-D-21-01271R1 | ✔ CP | ✔ DISK

$$\sup_{0<t\le T}\left|\int_{\xi t}^{t}(t-\tau)^{-\alpha-1}(V(x(t))-V(x(\tau)))d\tau\right|\to 0. \qquad (11)$$

Using Theorem 2.2 of [36] and (7), (11) gives $\exists D_C^\alpha V(x(t))\in C[0,T]$ and

$$\begin{aligned} D_C^\alpha(V(x(t)))(0) &= 2\Big(x(0),Pv(0)\Big),\\ D_C^\alpha(V(x(t))) &= \frac{V(x(t))-V(x(0))}{t^\alpha\Gamma(1-\alpha)}\\ &+\frac{\alpha}{\Gamma(1-\alpha)}\int_0^t\frac{V(x(t))-V(x(\tau))}{(t-\tau)^{\alpha+1}}d\tau,\ t\in(0,T]. \end{aligned} \qquad (12)$$

Besides we have $D_C^\alpha x\in C[0,T]$ and

$$\begin{aligned} (D_C^\alpha x)(0) &=\Gamma(\alpha+1)\frac{u(0)}{\Gamma(\alpha+1)}=u(0),\\ (D_C^\alpha x)(t) &=\frac{1}{\Gamma(1-\alpha)}\\ &\Big(\frac{x(t)-x(0)}{t^\alpha}+\frac{\alpha}{\Gamma(1-\alpha)}\int_0^t\frac{x(t)-x(\tau)}{(t-\tau)^{1+\alpha}}d\tau\Big). \end{aligned} \qquad (13)$$

The identities (12) and (13) lead to $D_C^\alpha(V(x(t)))-2(x(t),PD_C^\alpha x(t))=0, t=0$ and for $t\in(0,T]$ to

$$\begin{aligned} &D_C^\alpha(V(x(t)))-2(x(t),PD_C^\alpha x(t))\\ &=-\frac{V(x(t)-x(0))}{t^\alpha\Gamma(1-\alpha)}-\frac{\alpha}{\Gamma(1-\alpha)}\int_0^t\frac{V(x(t)-x(\tau))}{(t-\tau)^{\alpha+1}}d\tau\\ &\le 0, \end{aligned}$$

which completes the proof of Lemma 4.

To finish the theorem's proof, denoting

$$\xi(t)=[x(t),\ f(\cdot),\ g(\cdot)]^\top, f(\cdot)=f(x(t)),\ g(\cdot)=g(x(t-d(t))),$$

we obtain, by using Lemma 4, that

$$\begin{aligned} D_C^\alpha V(x(t)) &\le 2x(t)^\top PD_C^\alpha x(t)\\ &=2x(t)^\top P\Big(-Mx(t)+Ff(x(t))+Gg(\cdot)\Big)\\ &\le 2x(t)^\top P\Big(-Mx(t)+Ff(x(t))+Gg(\cdot)\Big)\\ &\quad-\beta f(\cdot)^\top f(\cdot)-\gamma g(\cdot)^\top Pg(\cdot)+\beta\max_i l_i^2x(t)^\top x(t)\\ &\quad-d_2x(t)^\top Px(t)+d_2V(x(t))+\gamma g(\cdot)^\top Pg(\cdot)\\ &=\xi(t)^\top[E_{ij}]_{3\times3}\xi(t)+d_2V(x(t))+\gamma g(\cdot)^\top Pg(\cdot)\\ &\le d_2V(x(t))+\gamma g(\cdot)^\top Pg(\cdot), \end{aligned} \qquad (14)$$

because of $\|f(\cdot)\|^2\le\max_i l_i^2\,\|x(t)\|^2$, and $[E_{ij}]_{3\times3}<0$ (by the condition (4)). Let

$$U(t)=D_C^\alpha V(x(t))-d_2V(x(t)),\ t\ge 0. \qquad (15)$$

Using the Laplace transform (by Lemma 1-(i)) to the both sides of (15) gives

$$\begin{aligned} \mathcal{L}[U(t)](s) &= s^\alpha\mathcal{L}[V(x(t))](s)-s^{\alpha-1}V(x(0))\\ &\quad-d_2\mathcal{L}[V(x(t))](s), \end{aligned}$$

equivalently

$$\begin{aligned} \mathcal{L}[V(x(t))](s) &= (s^\alpha-d_2)^{-1}s^{\alpha-1}V(x(0))\\ &\quad+(s^\alpha-d_2)^{-1}\mathcal{L}[U(t)](s). \end{aligned}$$

Applying Lemma 1 -(ii), (iii), we obtain that

$$\begin{aligned} \mathcal{L}\Big[V(x(0))E_\alpha(d_2t^\alpha)\Big](s) &=(s^\alpha-d_2)^{-1}s^{\alpha-1}V(x(0))\\ \mathcal{L}\Big[t^{\alpha-1}E_{\alpha,\alpha}(d_2t^\alpha)*U(t)\Big](s) &=(s^\alpha-d_2)^{-1}\mathcal{L}[U(t)](s), \end{aligned}$$

hence

$$\begin{aligned} &\mathcal{L}[V(x(t))](s)\\ &=\mathcal{L}\Big[V(x(0))E_\alpha(d_2t^\alpha)+t^{\alpha-1}E_{\alpha,\alpha}(d_2t^\alpha)*U(t)\Big](s). \end{aligned}$$

Taking the inverse Laplace transform to the derived equation gives

$$\begin{aligned} V(x(t)) &= V(x(0))E_\alpha(d_2t^\alpha)\\ &+\int_0^t\frac{U(s)}{(t-s)^{1-\alpha}}E_{\alpha,\alpha}(d_2(t-s)^\alpha)ds. \end{aligned} \qquad (16)$$

Using (14) and the inequality $\mathbb{I}_n\le P\le 2\mathbb{I}_n$, we have

$$\begin{aligned} U(t) &\le\gamma g(\cdot)^\top Pg(\cdot)\le 2\gamma g(\cdot)^\top g(\cdot)\\ &\le 2\gamma\max_i[k_i]^2\sum_{i=1}^n|x_i(t-d(t))|^2\\ &\le d_2x(t-d(t))^\top Px(t-d(t))=d_2V(x(t-d(t))), \end{aligned}$$

then

$$\sup_{s\in[0,t]}U(s)\le h_2\sup_{\theta\in[-d_2,t-d_1]}V(x(\theta)). \qquad (17)$$

From (16) and (17) it gives

$$\begin{aligned} V(x(t)) &\le V(x(0))E_\alpha(d_2t^\alpha)+\sup_{s\in[0,t]}U(s)\\ &\quad\int_0^t\frac{E_{\alpha,\alpha}(d_2(t-s)^\alpha)}{(t-s)^{1-\alpha}}ds\\ &\le V(x(0))E_\alpha(d_2t^\alpha)+(E_\alpha(d_2t^\alpha)-1)\\ &\quad\sup_{\theta\in[-d_2,t-d_1]}V(x(\theta)), \end{aligned}$$

Moreover, we have

Journal : Large 521 | Dispatch : 25-7-2021 | Pages : 8
Article No. : 6339 | LE | TYPESET
MS Code : NCAA-D-21-01271R1 | CP | DISK

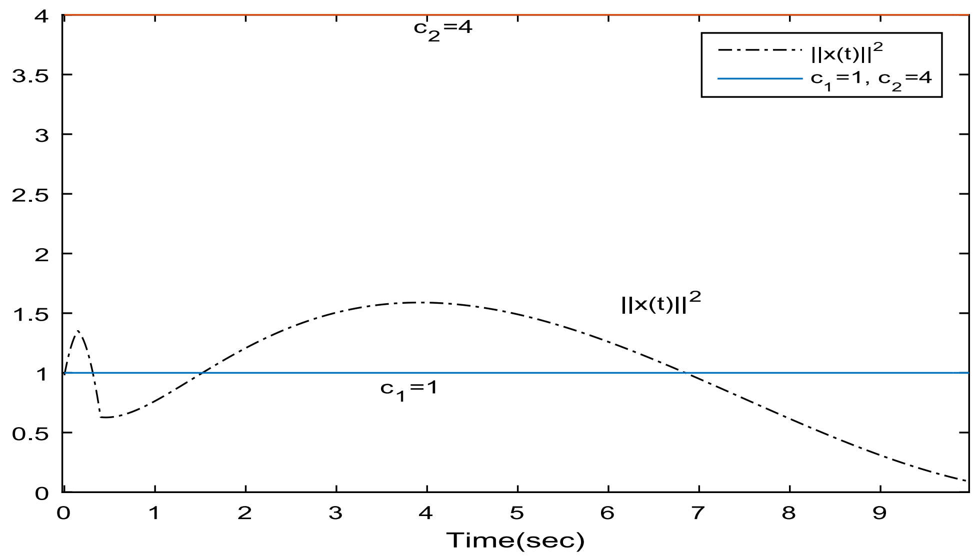

Fig. 1 Time history of $\|x(t)\|^2$ of the system with $\alpha = 0.5$

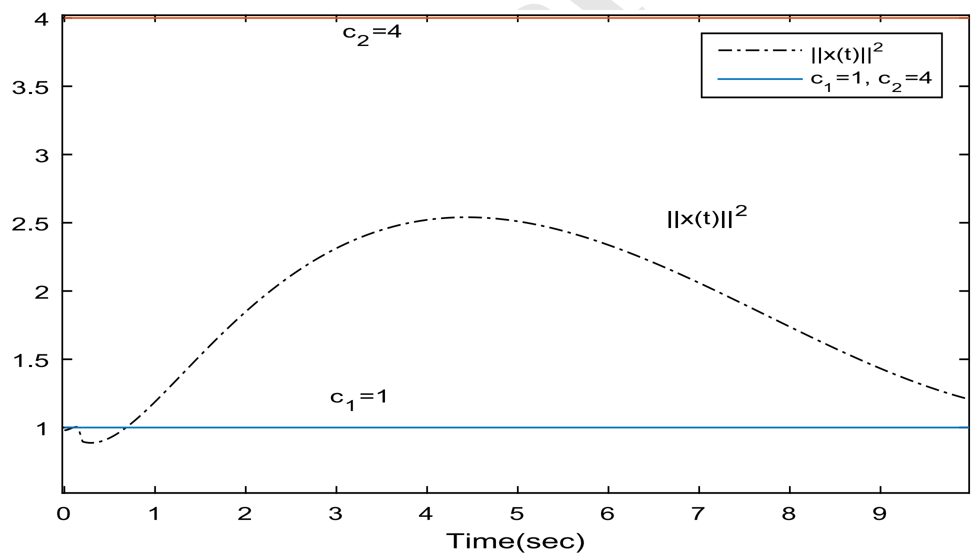

Fig. 2 Time history of $\|x(t)\|^2$ of the system with $\alpha = 0.6$

$$\begin{aligned}&\sup_{\theta\in[-d_2,t]} V(x(\theta)) \le E_\alpha(d_2T^\alpha)V(x(0))\\ &+[E_\alpha(d_2T^\alpha)-1]\sup_{\theta\in[-d_2,t-d_1]} V(x(\theta)).\end{aligned} \tag{18}$$

Applying Lemma 3 with $S(t) = \sup_{\theta\in[-d_2,t]} V(x(\theta)), a = E_\alpha(d_2T^\alpha)$, $b = E_\alpha(d_2T^\alpha) - 1$, and from (18) it follows that

$$\sup_{\theta\in[-d_2,t]} V(x(\theta)) \le q \sup_{\theta\in[-d_2,0]} V(x(\theta)) \le q\lambda_{max}(P)\|\phi\|^2, \tag{19}$$

, where $q = E_\alpha(d_2T^\alpha)\sum_{j=0}^{[T/d_1]+1}(E_\alpha(d_2T^\alpha)-1)^j$. For $t \in [0, T]$, the conditions (5) and (19) show that

$$\begin{aligned}\|x(t)\|^2 &\le \frac{x(t)^\top Px(t)}{\lambda_{min}(P)} \le \frac{\sup_{\theta\in[-d_2,t]} V(x(\theta))}{\lambda_{min}(P)}\\ &\le q\frac{\lambda_{max}(P)}{\lambda_{min}(P)}\|\phi\|^2 \le q\frac{\lambda_{max}(P)}{\lambda_{min}(P)}c_1 \le c_2,\end{aligned}$$

which shows that system (1) is FTS with respect to (c_1, c_2, T).

Remark 1 Note that the numbers c_1, c_2, do not involve in the LMI (4), we find the solutions P, β by solving LMI (4) and the condition (5) can be easily verified.

Remark 2 Theorem 1 proposed delay-dependent sufficient conditions for finite-time stability of FONNs with interval time-varying delay, which is a non-differentiable function, extends some existing results obtained in [23, 30–33], where the time delay is assumed to be differentiable.

Journal : Large 521 | Dispatch : 25-7-2021 | Pages : 8
Article No. : 6339 | LE | TYPESET
MS Code : NCAA-D-21-01271R1 | CP | DISK

Moreover, for the case fractional derivative order $\alpha = 1$, system (1) is reduced to normal fractional-order neural networks with time-varying delay and some existing results on FTS of such systems obtained in [4, 34, 37–39] can be derived from Theorem 1.

Remark 3 It should be pointed out that the advantage of our paper was proposing an approach based on the Laplace transform combining with the inf–sup method to study stability of FONNs with interval time-varying delay without using the fractional Lyapunov stability theorem.

Example 1 Consider FONNs (1) with the following system parameters

$$\alpha = 0.5,\ d(t) = 0.1 + 0.05|\sin^2(t)|,$$
$$M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix},\ B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix},$$

the neuron activation functions $f, g : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$f(x) = (f_1(x_1), f_2(x_2))^\top,\ g(x) = (g_1(x_1), g_2(x_2))^\top,$$
$$f_1(t) = f_2(t) = g_1(t) = g_2(t) = 0.08\frac{t}{1+t^2},$$

for all $t \in \mathbb{R},\ (x_1, x_2) \in \mathbb{R}^2$.

It can be shown that $0 < d_1 = 0.1 \le d(t) \le d_2 = 0.15$, $f(0) = g(0) = 0$, and the neuron activation functions satisfying the Lipschitz conditions (3) with $l_1 = l_2 = k_1 = k_2 = 0.1$. Since the delay function $d(t)$ is non-differentiable, the method used in [20, 30–33] cannot be applied. We use the LMI algorithm in MATLAB [40] to find solutions of (4) as

$$P = \begin{bmatrix} 1.7413 & 0.1105 \\ 0.1105 & 1.7544 \end{bmatrix},\ \beta = 5.8115.$$

In this case, it can be computed that

$$\gamma = 7.5,\ \lambda_{max}(P) = 1.8586,\ \lambda_{min}(P) = 1.6371.$$

For $c_1 = 1,\ c_2 = 4,\ T = 10$, we can check the condition (5) as

$$E_\alpha(d_2 T^\alpha) \sum_{j=0}^{[T/d_1]+1} (E_\alpha(d_2 T^\alpha) - 1)^j \frac{\lambda_{max}(P)}{\lambda_{min}(P)} c_1 = 3.9939 < 4$$

Hence, by Theorem 1, the system (1) is FTS with respect to (1, 4, 10). Figure 1 and Figure 2 demonstrate the time history $\|x(t)\|^2$ of the system with initial condition $\phi(t) = [0.65, 0.65],\ t \in [-0.15,\ 0]$ and $\alpha = 0.5$, and $\alpha = 0.6$,, respectively.

4 Conclusions

In this paper, the finite-time stability problem for a class of FONNs with interval time-varying delay has been addressed. Based on a novel analytical approach, delay-dependent sufficient conditions for FTS are proposed. The conditions are presented in the form of a tractable LMI and Mittag–Leffler functions. Finite-time stability analysis of FONNs with unbounded time-varying delay may be interesting topics to study in the future, and an extension of this study to non-autonomous FONNs with delays is an open problem.

Acknowledgements This paper was written when the authors were studying at the Vietnam Institute for Advanced Study in Mathematics (VIASM). We sincerely thank the Institute for support and hospitality. The research of N.T. Thanh and V.N. Phat is supported by National Foundation for Science and Technology Development (No. 101.01-2021.01). The research of P. Niamsup is supported by the Chiang Mai University, Thailand. The authors wish to thank the associate editor and anonymous reviewers for valuable comments and suggestions, which allowed us to improve the paper.

Declarations

Conflict of interest The authors declare that no potential conflict of interest to be reported to this work.

References

1. Forti F, Tesi A (1995) New conditions for global stability of neural networks with application to linear and quadratic programming problems, IEEE Trans Circuits Syst I Funda Theory Appl. 42(7), 354–366
2. Arik S (2002) An analysis of global asymptotic stability of delayed cellular neural networks. IEEE Trans. on Neural Netw. 3(5), 1239–1242
3. Gupta D, Cohn LF (2012) Intelligent transportation systems and NO2 emissions: Predictive modeling approach using artificial neural networks. J. Infrastruct. Syst. 18(2), 113–118
4. Phat VN, Fernando T, Trinh H (2014) Observer-based control for time-varying delay neural networks with nonlinear observation. Neural Computing and Applications 24:1639–1645
5. Abdelsalam SI, Velasco-Hernandez JX, Zaher A.Z. (2012) Electro-magnetically modulated self-propulsion of swimming sperms via cervical canal. Biomechanics and Modeling in Mechanobiology 20:861–878
6. AbdelsalamS I, Bhatti MM (2020) Anomalous reactivity of thermo-bioconvective nanofluid towards oxytactic microorganisms. Applied Mathematics and Mechanics 41:711–724
7. Kilbas AA, Srivastava H, Trujillo J (2006) Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam
8. Bhatti MM, Alamri SZ, Ellahi R, Abdelsalam S.I. (2021) Intra-uterine particle-fluid motion through a compliant asymmetric tapered channel with heat transfer. Journal of Thermal Analysis and Calorimetry 144:2259–2267
9. Elmaboud Y.A., Abdelsalam S.I., (2019), DC/AC magnetohydro dynamic-micropump of a generalized Burger's fluid in an annulus, Physica Scripta, 94(11): 115209

Journal : Large 521 | Dispatch : 25-7-2021 | Pages : 8
Article No. : 6339 | ☐ LE | ☐ TYPESET
MS Code : NCAA-D-21-01271R1 | ☑ CP | ☑ DISK

10. Lundstrom B, Higgs M, Spain W, Fairhall A (2008) Fractional differentiation by neocortical pyramidal neurons. Nat. Neurosci. 11:1335–1342
11. Zhang S, Yu Y, Wang H (2015) Mittag-Leffler stability of fractional-order Hopfield neural networks. Nonlinear Anal. Hybrid Syst. 16:104–121
12. Chen L, Liu C, Wu R et al. (2016) Finite-time stability criteria for a class of fractional-order neural networks with delay. Neural Computing and Applications 27:549–556
13. Liu S, Yu Y, Zhang S (2019) Robust synchronization of memristor-based fractional-order Hopfield neural networks with parameter uncertainties. Neural Computing and Applications 31:3533–3542
14. Baleanu D, Asad JH, Petras I (2012) Fractional-order two-electric pendulum. Romanian Reports in Physics 64(4), 907–914
15. Baleanu D, Petras I, Asad JH, Velasco MP (2012) Fractional Pais-Uhlenbeck oscillator. International Journal of Theoretical Physics 51(4), 1253–1258
16. Baleanu D, Asad JH, Petras I (2015) Numerical solution of the fractional Euler-Lagrange's equations of a thin elastica model. Nonlinear Dynamics 81(1–2), 97–102
17. Dorato P (1961) Short time stability in linear time-varying systems. Proceedings of IRE International Convention Record 4:83–87
18. Amato F, Ambrosino R, Ariola M, Cosentino C (2014) Finite-Time Stability and Control Lecture Notes in Control and Information Sciences, Springer, New York
19. Cai M, Xiang Z (2015) Adaptive fuzzy finite-time control for a class of switched nonlinear systems with unknown control coefficients. Neurocomputing 162:105–115
20. Fan Y, Li Y (2020) Adaptive fuzzy finite-time optimal control for switched nonlinear systems. Appl Meth Optimal Contr. doi: https://doi.org/10.1002/oca.2623
21. Ruan Y., Huang T. (2020), Finite-time control for nonlinear systems with time-varying delay and exogenous disturbance, Symmetry, 12(3): 447
22. Lazarevi MP, Spasi AM (2009) Finite-time stability analysis of fractional order time-delay systems: Gronwall's approach. Math. Comput. Model. 49:475–481
23. Phat VN, Thanh NT (2018) New criteria for finite-time stability of nonlinear fractional-order delay systems: A Gronwall inequality approach. Appl. Math. Letters 83:169–175
24. Ye H, Gao J, Ding Y (2007) A generalized Gronwall inequality and its application to a fractional differential equation. J. Math. Anal. Appl. 328:1075–1081
25. Yang X, Song Q, Liu Y, Zhao Z (2015) Finite-time stability analysis of fractional-order neural networks with delay. Neurocomputing 152:19–26
26. Chen L, Liu C, Wu R, He Y, Chai Y (2016) Finite-time stability criteria for a class of fractional-order neural networks with delay. Neural Computing and Applications 27:549–556
27. Wu RC, Lu YF, Chen LP (2015) Finite-time stability of fractional delayed neural networks. Neurocomputing 149:700–707
28. Xu C, Li P (2019) On finite-time stability for fractional-order neural networks with proportional delays. Neural Processing Letters 50:1241–1256
29. Rajivganthi C, Rihan FA, Lakshmanan S, Muthukumar P (2018) Finite-time stability analysis for fractional-order Cohen-Grossberg BAM neural networks with time delays. Neural Computing and Applications 29:1309–1320
30. Hua C, Zhang T, Li Y, Guan X (2016) Robust output feedback control for fractional-order nonlinear systems with time-varying delays. IEEE/CAA J. Auto. Sinica 3:47–482
31. Zhang H, Ye R, Cao J, Ahmed A et al. (2018) Lyapunov functional approach to stability analysis of Riemann-Liouville fractional neural networks with time-varying delays, Asian. J. Control 20:1–14
32. Zhang H, Ye R, Liu S et al. (2018) LMI-based approach to stability analysis for fractional-order neural networks with discrete and distributed delays. Int. J. Syst. Science 49:537–545
33. Zhangand F, Zeng Z (2020) Multistability of fractional-order neural networks with unbounded time-varying delays. Neural Netw Lean Syst IEEE Trans https://doi.org/10.1109/TNNLS.2020.2977994
34. Thanh NT, Niamsup P, Phat VN (2020) New finite-time stability analysis of singular fractional differential equations with time-varying delay. Frac. Calcul. Anal. Appl. 23:504–517
35. Sau NH, Hong DT, Huyen NT, et al. (2021) Delay-dependent and order-dependent H_∞ control for fractional-order neural networks with time-varying delay. Equ Dyn Syst Diff. doi: https://doi.org/10.1007/s12591-020-00559-z
36. Vainikko G (2016) Which functions are fractionally differentiable. Zeitschrift fuer Analysis und Ihre Anwendungen 35:465–48
37. Cheng J, Zhong S, Zhong Q, Zhu H, Du YH (2014) Finite-time boundedness of state estimation for neural networks with time-varying delays. Neurocomputing 129:257–264
38. Prasertsang P, Botmart T (2021) Improvement of finite-time stability for delayed neural networks via a new Lyapunov-Krasovskii functional. AIMS Mathematics 6(1), 998–1023
39. Saravanan S., Ali M. S. (2018), Improved results on finite-time stability analysis of neural networks with time-varying delays, J. Dyn. Sys. Meas. Control., 140(10): 101003
40. Gahinet P, Nemirovskii A, Laub AJ, Chilali M (1985) LMI Control Toolbox For use with Matlab. The MathWorks Inc, Massachusetts

Journal : Large 521 | Dispatch : 25-7-2021 | Pages : 8
Article No. : 6339 | ☐ LE | ☐ TYPESET
MS Code : NCAA-D-21-01271R1 | ☑ CP | ☑ DISK