# MỤC LỤC

## Phần A. Động lực học và điều khiển robot

# Linearization of the motion equations of rigid-flexible manipulators

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**Abstract**
The paper deals with the application of a computer algebraic approach for linearization of the motion equations for flexible manipulators. It is used a Taylor series expansion of the nonlinear equations of motion to obtain the desired basic motion of the system. Based on the proposed method, the linearization of motion equations of a rigid-flexible translation and rotation two-link manipulator (manipulator T-R) and of a rigid-flexible rotation and rotation two-link manipulator (manipulator R-R) has been studied.
***Keywords:*** Multibody system, Flexible manipulators, Linearization, The motion equations, fundamental motion.

## 1. Introduction

Multibody system dynamics has experienced a significant growth over the last decades, boosted by advances in computer architecture and software. The position of a tree-structured multibody system is most often represented herein by an array $\mathbf{s} = [s_1\, s_2 \ldots s_n]^T$ of generalized coordinates. The velocity of the system is described as the array of generalized velocities $\dot{\mathbf{s}} = [\dot{s}_1\, \dot{s}_2 \ldots \dot{s}_n]^T$. The set of generalized coordinates and generalized velocities can be selected in a multitude of ways. The equations of motion for a general rigid body system can usually be written in the following form [1-5]

$$\mathbf{M}(\mathbf{s})\ddot{\mathbf{s}} + \mathbf{C}(\mathbf{s},\dot{\mathbf{s}})\dot{\mathbf{s}} + \mathbf{g}(\mathbf{s}) = \boldsymbol{\tau}(t) \tag{1}$$

In this equation, $\mathbf{s}, \dot{\mathbf{s}}$ and $\ddot{\mathbf{s}}$ are vectors of generalized position, velocity, and acceleration variables, respectively. The differential equations of motion (1) are, in general, a system of related nonlinear differential equations.

Using the floating frame of reference approach by the derivation of motion equations of flexible manipulators [6], we also get a system of motion differential equations of the form (1).

Due to the complexity of the system of differential equations of motion (1), it is very difficult for us to find the solution by analytical methods. It is common to determine the solution of system (1) by numerical methods. Using the numerical methods to solve the system of the differential equation (1) we need to have 2n the initial conditions

$$\mathbf{s}(t_0) = \mathbf{s}_0\,,\ \dot{\mathbf{s}}(t_0) = \dot{\mathbf{s}}_0 \tag{2}$$

In this paper, the linearization problem of nonlinear motion equations of flexible manipulators in the vicinity of a fundamental motion is addressed. Taylor series expansion of nonlinear motion equations around the desired fundamental motion of the flexible manipulator is included

The paper is organized as follows: The basis of the linearization algorithm is presented in Section 2. Sections 3 and 4 present the linearization of motion equations of the rigid-flexible two-link T-R and R-R around the fundamental motion. The final part is the conclusion.

## 2. Derivation of the linearization procedure

Many technical systems mostly work in the vicinity of a desired movement, which is usually called "program movement", "basic movement", etc., depending on the problem.

In the dynamics of robot manipulators with flexible links, the motion of the flexible manipulator $\mathbf{q}(t)$ is usually in the vicinity of a motion $\mathbf{q}_R(t)$ of a corresponding rigid manipulator.

Motion equations of flexible robot manipulators (1) can be rewritten in the following form

$$\mathbf{M}(\mathbf{s})\ddot{\mathbf{s}} = \mathbf{p}(\dot{\mathbf{s}},\mathbf{s},\boldsymbol{\tau},t) \tag{3}$$

where

$$\mathbf{p}(\dot{\mathbf{s}},\mathbf{s},\boldsymbol{\tau},t) = \boldsymbol{\tau}(t) - \mathbf{C}(\mathbf{s},\dot{\mathbf{s}})\dot{\mathbf{s}} - \mathbf{g}(\mathbf{s}) \tag{4}$$

By introducing

$$\mathbf{s}(t) = \mathbf{s}^{\mathbf{R}}(t) + \Delta\mathbf{s}(t) = \mathbf{s}^{\mathbf{R}}(t) + \mathbf{y}(t)\,. \tag{5}$$

$$\dot{\mathbf{s}}(t) = \dot{\mathbf{s}}^{\mathbf{R}}(t) + \Delta\dot{\mathbf{s}}(t) = \dot{\mathbf{s}}^{\mathbf{R}}(t) + \dot{\mathbf{y}}(t)\,. \tag{6}$$

$$\ddot{\mathbf{s}}(t) = \ddot{\mathbf{s}}^{\mathbf{R}}(t) + \Delta\ddot{\mathbf{s}}(t) = \ddot{\mathbf{s}}^{\mathbf{R}}(t) + \ddot{\mathbf{y}}(t)\,. \tag{7}$$

$$\boldsymbol{\tau}(t) = \boldsymbol{\tau}^R + \Delta\boldsymbol{\tau}\,. \tag{8}$$

where $\mathbf{s}^{\mathbf{R}}(t)$ is the fundamental motion of the robot manipulator, $\boldsymbol{\tau}^R(t)$ is the torque, in which the elastic link is considered as rigid

$$\mathbf{s}^R(t) = \begin{bmatrix} \mathbf{q}_a^R(t) \\ 0 \end{bmatrix}, \dot{\mathbf{s}}^R(t) = \begin{bmatrix} \dot{\mathbf{q}}_a^R(t) \\ 0 \end{bmatrix}, \ddot{\mathbf{s}}^R(t) = \begin{bmatrix} \ddot{\mathbf{q}}_a^R(t) \\ 0 \end{bmatrix} \tag{9}$$

To simplify the implementation of intermediate transformations, we introduce the symbols

$$\mathbf{f}(\ddot{\mathbf{s}},\mathbf{s}) = \mathbf{M}(\mathbf{s})\ddot{\mathbf{s}}\,. \tag{10}$$

where $\mathbf{f} \in \Re^{n\times 1}$, $\mathbf{p} \in \Re^{n\times 1}$.

The following equations are obtained by means of Taylor series expansion of $\mathbf{f}(\ddot{\mathbf{s}},\mathbf{s})$ and $\mathbf{p}(\dot{\mathbf{s}},\mathbf{s},\tau,t)$ around the desired fundamental motion $\mathbf{s}_R, \dot{\mathbf{s}}_R, \ddot{\mathbf{s}}_R, \boldsymbol{\tau}^R$:

$$\mathbf{f}(\ddot{\mathbf{s}},\mathbf{s}) = \mathbf{f}(\ddot{\mathbf{s}}_R + \ddot{\mathbf{y}}, \mathbf{s}_R + \mathbf{y})$$
$$= \mathbf{f}(\ddot{\mathbf{s}}_R, \mathbf{s}_R) + \left.\frac{\partial \mathbf{f}_1}{\partial \ddot{\mathbf{s}}}\right|_R \ddot{\mathbf{y}} + \left.\frac{\partial \mathbf{f}_1}{\partial \mathbf{s}}\right|_R \mathbf{y} + nonline.\ terms, \quad (11)$$

$$\mathbf{p}(\dot{\mathbf{s}},\mathbf{s},\boldsymbol{\tau},t) = \mathbf{p}(\dot{\mathbf{s}}_R + \dot{\mathbf{y}}, \mathbf{s}_R + \mathbf{y}, \boldsymbol{\tau}^R + \Delta\boldsymbol{\tau}, t) =$$
$$= \mathbf{p}_1(\dot{\mathbf{s}}_R, \mathbf{s}_R, \boldsymbol{\tau}^R, t) + \left.\frac{\partial \mathbf{p}_1}{\partial \dot{\mathbf{s}}}\right|_R \dot{\mathbf{y}} + \left.\frac{\partial \mathbf{p}_1}{\partial \mathbf{s}}\right|_R \mathbf{y} \quad (12)$$
$$+ \left.\frac{\partial \mathbf{p}_1}{\partial \boldsymbol{\tau}}\right|_R \Delta\boldsymbol{\tau} + nonlinear\ terms.$$

Substituting Eqs. (11) and (12) into Eq. (3) and neglecting nonlinear terms, we obtain the linearized differential equations that describe vibrations of considered system as follows

$$\mathbf{M}_L(t)\ddot{\mathbf{y}} + \mathbf{C}_L(t)\dot{\mathbf{y}} + \mathbf{K}_L(t)\mathbf{y} = \mathbf{h}_L(t) \quad (13)$$

The matrices in Eq. (13) have the following form

$$\mathbf{M}_L(t) = \left.\frac{\partial \mathbf{f}}{\partial \ddot{\mathbf{s}}}\right|_R \quad \mathbf{C}_L(t) = -\left.\frac{\partial \mathbf{p}}{\partial \dot{\mathbf{s}}}\right|_R, \quad (14)$$

$$\mathbf{K}_L(t) = \left( \left.\frac{\partial \mathbf{f}}{\partial \mathbf{s}}\right|_R - \left.\frac{\partial \mathbf{p}}{\partial \mathbf{s}}\right|_R \right), \quad (15)$$

$$\mathbf{h}_L(t) = \mathbf{p}(\dot{\mathbf{s}}_R, \mathbf{s}_R, \boldsymbol{\tau}^R, t) - \mathbf{f}(\ddot{\mathbf{s}}_R, \mathbf{s}_R) + \left.\frac{\partial \mathbf{p}_1}{\partial \boldsymbol{\tau}}\right|_R \Delta\boldsymbol{\tau}. \quad (16)$$

## 3. Linearization of the motion equations of a rigid-flexible two-link manipulator T-R

### 3.1. Derivation of equations of motion for flexible manipulators R-T using the floating frame of the reference approach

Consider the motion of a two-link rigid-flexible manipulator T-R shown in Fig.1, where the rigid link AB assumed to be uniform.

To describe the kinematics, the position of point P on the flexible beam is given as, in which $w(x,t)$ is the transverse deformation of the beam DE

$$x_P = l_1 + (r+x)\cos q_{a2} - w\sin q_{a2}. \quad (17)$$
$$y_P = q_{a1} + (r+x)\sin q_{a2} + w\cos q_{a2} \quad (18)$$

Differentiation of Eq. (18) yields

$$v_P^2 = \dot{x}_P^2 + \dot{y}_P^2 = \dot{q}_{a1}^2 + \dot{w}^2 + [(r+x)^2 + w^2]\dot{q}_{a2}^2$$
$$+ 2(r+x)\dot{w}\dot{q}_{a2} + 2(r+x)\dot{q}_{a1}\dot{q}_{a2}\cos q_{a2}$$
$$+ 2\dot{q}_{a1}\dot{w}\cos q_{a2} - 2w\dot{q}_{a1}\dot{q}_{a2}\sin q_{a2}. \quad (19)$$

The Euler-Bernoulli beam theory and Ritz-Galerkin method are applied to the flexible manipulator with assuming that the deformation in the longitudinal direction is negligibly small. Let the transverse deformation of the beam be written as

$$w(x,t) = \sum_{i=1}^{N} X_i(x) q_{ei}(t), \; w_E = \sum_{i=1}^{N} X_i(l_2) q_{ei}(t), \quad (20)$$

where $q_{ei}(t)$ denotes the modal coordinates of transverse deformation, $X_i(x)$ the mode shapes of transverse deformation of a clamped- free beam. The mode shapes are given as [7]

$$X_i(x) = \cos(\beta_i x) - \cosh(\beta_i x)$$
$$+ \frac{\cos\beta_i l + \cosh\beta_i l}{\sin\beta_i l + \sinh\beta_i l}\left(-\sin(\beta_i x) + \sinh(\beta_i x)\right) \quad (21)$$

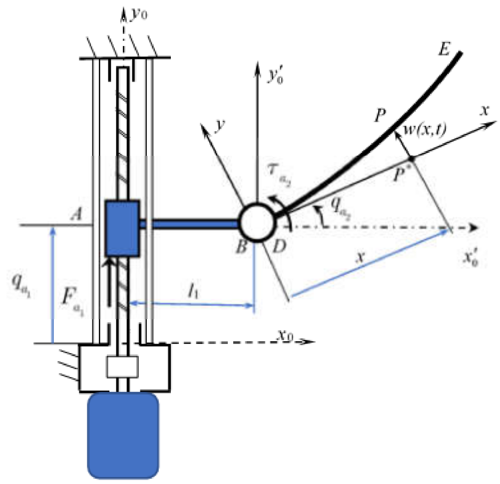


**Fig. 1.** Two-link rigid-flexible manipulator T-R

The kinetic energy of the flexible manipulator shown in Fig. 1 is given by

$$T = T_1 + T_2 + T_B \quad (20)$$

Where $T_1$ is the expression of kinetic energy of solid link 1, $T_2$ is the expression of kinetic energy of elastic link DE, $T_B$ is the kinetic energy of the disc B.

$$T_1 = \frac{1}{2} m_1 \dot{q}_{a1}^2. \quad (21)$$

$$T_2 = \frac{1}{2}\mu\int_0^{l_2} [\dot{q}_{a1}^2 + \dot{w}^2 + [(r+x)^2 + w^2]\dot{q}_{a2}^2$$
$$+ 2(r+x)\dot{w}\dot{q}_{a2} + 2(r+x)\dot{q}_{a1}\dot{q}_{a2}\cos q_{a2} \quad (22)$$
$$+ 2\dot{q}_{a1}\dot{w}\cos q_{a2} - 2w\dot{q}_{a1}\dot{q}_{a2}\sin q_{a2}]dx$$

$$T_B = \frac{1}{2} m_B \dot{q}_{a1}^2 + \frac{1}{2} J_B \dot{q}_{a2}^2. \quad (23)$$

Thus, the expression for the kinetic energy of the manipulator has the form

$$T = \frac{1}{2} m_1 \dot{q}_{a1}^2 + \frac{1}{2} m_B \dot{q}_{a1}^2 + \frac{1}{2} J_B \dot{q}_{a2}^2 + \frac{1}{2}\mu l_2 \dot{q}_{a1}^2$$
$$+ \frac{1}{2}\mu(r^2 l_2 + r l_2^2 + \frac{l_2^3}{3})\dot{q}_{a2}^2 + \frac{1}{2}\mu(2r l_2 + l_2^2)\dot{q}_{a1}\dot{q}_{a2}\cos q_{a2} \quad (24)$$
$$+ \frac{1}{2}\mu\int_0^{l_2} [\dot{w}^2 + w^2\dot{q}_{a2}^2 + 2(r+x)\dot{w}\dot{q}_{a2}$$
$$+ 2\dot{q}_{a1}\dot{w}\cos q_{a2} - 2w\dot{q}_{a1}\dot{q}_{a2}\sin q_{a2}]dx$$

By substituting $m_2 = \mu l_2$ into Eq. (24), we have

$$
\begin{aligned}
T &= \frac{1}{2}(m_1+m_2+m_B)\dot{q}_{a1}^2 + (\frac{1}{2}J_B + \frac{1}{2}m_2 r^2 + \\
&+ \frac{1}{2}m_2 r l_2 + \frac{1}{6}m_2 l_2^2)\dot{q}_{a2}^2 + (\frac{1}{2}m_2 l_2 + m_2 r)\dot{q}_{a1}\dot{q}_{a2}\cos q_{a2} \\
&+ \frac{1}{2}\mu\int_0^{l_2}[\dot{w}^2 + w^2\dot{q}_{a2}^2 + 2(r+x)\dot{w}\dot{q}_{a2} \\
&+ 2\dot{q}_{a1}\dot{w}\cos q_{a2} - 2w\dot{q}_{a1}\dot{q}_{a2}\sin q_{a2}]dx
\end{aligned} \tag{25}
$$

The elastic potential energy of the beam DE to transverse bending is calculated by the formula [8]

$$
\Pi_1 = \frac{1}{2}\int_0^{l_2}\left[EI\left(\frac{\partial^2 w}{\partial x^2}\right)^2\right]dx . \tag{26}
$$

Where EI is the beam flexural rigidity. Calculating the gravity potential energy, we have

$$
\begin{aligned}
\Pi &= (m_1+m_2+m_B)gq_{a1} + m_2 g(r+\frac{l_2}{2})\sin q_{a2} \\
&+ \mu g\cos q_{a2}\int_0^{l_2} w dx + \frac{1}{2}EI\int_0^{l_2}\left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx
\end{aligned} \tag{27}
$$

Using the expressions (18) and (19), the expression for kinetic energy (24) and the expression for potential energy (27) have the following form

$$
\begin{aligned}
T &= \frac{1}{2}(m_1+m_2+m_B)\dot{q}_{a1}^2 + (\frac{1}{2}J_B + \frac{1}{2}m_2 r^2 \\
&+ \frac{1}{2}m_2 r l_2 + \frac{1}{6}m_2 l_2^2)\dot{q}_{a1}^2 + (\frac{1}{2}m_2 l_2 + m_2 r)\dot{q}_{a1}\dot{q}_{a2}\cos q_{a2} \\
&+ \frac{1}{2}\mu\sum_{i=1}^{N}\sum_{j=1}^{N} n_{ij}\dot{q}_{ei}\dot{q}_{ej} + \frac{1}{2}\mu\dot{q}_{a_2}^2\sum_{i=1}^{N}\sum_{j=1}^{N} n_{ij}q_{ei}q_{ej} \\
&+ \mu r\dot{q}_{a2}\sum_{i=1}^{N} C_i\dot{q}_{ei} + \mu\dot{q}_{a2}\sum_{i=1}^{N} D_i\dot{q}_{ei} \\
&+ \mu\dot{q}_{a1}\cos q_{a2}\sum_{i=1}^{N} C_i\dot{q}_{ei} - \mu\dot{q}_{a1}\dot{q}_{a2}\sin q_{a2}\sum_{i=1}^{N} C_i q_{ei}
\end{aligned} \tag{28}
$$

$$
\begin{aligned}
\Pi &= (m_1+m_2+m_B)gq_{a1} + m_2 g(r+\frac{l_2}{2})\sin q_{a2} \\
&+ mg\cos q_{a2}\sum_{i=1}^{N} C_i q_{ei} + \frac{1}{2}EI\sum_{i=1}^{N}\sum_{j=1}^{N} h_{ij}q_{ei}q_{ej}
\end{aligned} \tag{29}
$$

where

$$
C_i = \int_0^{l_2} X_i dx \ ; \quad D_i = \int_0^{l_2} xX_i dx ; \quad n_{ij} = \int_0^{l_2} X_i X_j dx \tag{30}
$$

$$
h_{ij} = \int_0^{l_2} X_i'' X_j'' dx \tag{31}
$$

The generalized forces $Q_j^*$ are determined by the following expressions

$$
Q_{q_{a1}}^* = F_{a1} - F_{d1} ; \quad Q_{q_{a2}}^* = \tau_{a2} - M_{d2} \ ; \quad Q_{q_{ei}}^* = 0 . \tag{32}
$$

where

$$
F_{d1} = \alpha_1\dot{q}_{a1}, M_{d2} = \alpha_2\dot{q}_{a2} \tag{33}
$$

The Lagrange equations have the following form [4, 5]

$$
\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = -\frac{\partial \Pi}{\partial q_j} + Q_j^* , \text{ j=1,2,…,n} , \tag{34}
$$

where $q_j$ are the generalized coordinates which include rigid body coordinate $q_a$ as well as elastic modal $q_{ei}$, and $Q_j^*$ are generalized forces.

Substitution of Eqs. (28), (29) and (32) into Eq. (34) yields the equations of motion of the considered system as follows

$$
\begin{aligned}
&F_{d1} + (m_1+m_2+m_B)\ddot{q}_{a1} + (\frac{1}{2}m_2 l_2 + m_2 r)\ddot{q}_{a1}\cos q_{a2} \\
&-(\frac{1}{2}m_2 l_2 + m_2 r)\dot{q}_{a2}^2\sin q_{a2} + \mu\cos q_{a2}\sum_{i=1}^{N} C_i\ddot{q}_{ei} \\
&-2\mu\dot{q}_{a2}\sin q_{a2}\sum_{i=1}^{N} C_i\dot{q}_{ei} - \mu\ddot{q}_{a2}\sin q_{a2}\sum_{i=1}^{N} C_i q_{ei} \\
&-\mu\dot{q}_{a2}^2\cos q_{a2}\sum_{i=1}^{N} C_i q_{ei} = -(m_1+m_2+m_B)g + F_{a1}
\end{aligned} \tag{35}
$$

$$
\begin{aligned}
&\mathrm{M}_{d2} + [(\frac{1}{2}m_2 l_2 + m_2 r)\cos q_{a2} - \mu\sin q_{a2}\sum_{i=1}^{N} C_i q_{ei}]\ddot{q}_{a1} \\
&+(J_B + m_2 r^2 + m_2 r l_2 + \frac{m_2 l_2^2}{3})\ddot{q}_{a2} + \mu\ddot{q}_{a2}\sum_{i=1}^{N}\sum_{j=1}^{N} n_{ij}q_{ei}q_{ej} \\
&+\mu r\sum_{i=1}^{N} C_i\ddot{q}_{ei} + \mu\sum_{i=1}^{N} D_i\ddot{q}_{ei} + 2\mu\dot{q}_{a2}\sum_{i=1}^{N}\sum_{j=1}^{N} n_{ij}\dot{q}_{ei}q_{ej} \\
&= -m_2 g(r+\frac{l_2}{2})\cos q_{a2} + \mu g\sin q_{a2}\sum_{i=1}^{N} C_i q_{ei} + \tau_{a2}
\end{aligned} \tag{36}
$$

$$
\begin{aligned}
&\mu\sum_{j=1}^{N} m_{ij}\ddot{q}_{ei} + \mu r\ddot{q}_{a2}C_i + \mu D_i\ddot{q}_{a2} + \mu C_i\ddot{q}_{a1}\cos q_{a2} \\
&-\mu\dot{q}_{a2}^2\sum_{j=1}^{N} n_{ij}q_{ej} = -\mu g C_i\cos q_{a2} - EI\sum_{j=1}^{N} h_{ij}q_{ej}
\end{aligned} \tag{37}
$$

If we choose N = 1, the differential equations of the two-link rigid-flexible manipulator T-R have the following form

$$
\begin{aligned}
&F_{d1} + (m_1+m_2+m_B)\ddot{q}_{a1} + \mu\cos q_{a2}C_1\ddot{q}_{e1} \\
&+[(\frac{1}{2}m_2 l_2 + m_2 r)\cos q_{a2} - \mu C_1 q_{e2}\sin q_{a2}]\ddot{q}_{a2} \\
&-(\frac{1}{2}m_2 l_2 + m_2 r)\dot{q}_{a2}^2\sin q_{a2} - 2\mu\dot{q}_{a2}C_1\dot{q}_{e1}\sin q_{a2} \\
&-\mu\dot{q}_{a2}^2 C_1 q_{e1}\cos q_{a2} = -(m_1+m_2+m_B)g + F_{a1}
\end{aligned} \tag{38}
$$

$$
\begin{aligned}
&M_{d2} + [(\frac{m_2 l_2}{2} + m_2 r)\cos q_{a2} - \mu C_1 q_{e1}\sin q_{a2}]\ddot{q}_{a1} \\
&+(J_B + m_2 r^2 + m_2 r l_2 + \frac{m_2 l_2^2}{3} + \mu n_{11}q_{e1}^2)\ddot{q}_{a2} \\
&+(\mu r C_1 + \mu D_1)\ddot{q}_{e1} + 2\mu n_{11}\dot{q}_{a2}\dot{q}_{e1}q_{e1} \\
&= -m_2 g(r+\frac{l_2}{2})\cos q_{a2} + \mu g\sin q_{a2}C_1 q_{e1} + \tau_{a2}
\end{aligned} \tag{39}
$$

$$\mu C_1\ddot{q}_{a1}\cos q_{a2}+(\mu rC_1+\mu D_1)\ddot{q}_{a2}+\mu n_{11}\ddot{q}_{e1}-\mu\dot{q}_{a2}^2 n_{11}q_{e1}$$
$$=-\mu gC_1\cos q_{a2}-EIh_{11}q_{e1}\,. \quad (40)$$

### 3.2. Fundamental motion of the flexible manipulator

We consider the case of the manipulator shown in Fig. 1, that consists of the two rigid links, the fundamental motion of the corresponding flexible manipulator is the virtual rigid link motion of the link DE. From the virtual rigid link motion, the position of the point P on the link BE is given as

$$\begin{cases} x_P = l_1+(r+x)\cos q_{a2}^R \\ y_P = q_{a1}+(r+x)\sin q_{a2}^R \end{cases}. \quad (41)$$

The kinetic energy of the manipulator is given by

$$T=T_1+T_2+T_B \quad (42)$$

where

$$T_1=\frac{1}{2}m_1\dot{q}_{a1}^2 \quad (43)$$

$$T_B=\frac{1}{2}m_B(\dot{q}_{a1}^R)^2+\frac{1}{2}J_B(\dot{q}_{a2}^R)^2\,. \quad (44)$$

$$T_2=\frac{1}{2}m_2(\dot{q}_{a1}^R)^2+(\frac{1}{2}r^2+\frac{1}{2}rl_2+\frac{1}{6}l_2^2)m_2(\dot{q}_{a2}^R)^2 + (\frac{l_2}{2}+r)m_2\dot{q}_{a1}^R\dot{q}_{a2}^R\cos q_{a2}^R \quad (45)$$

By substituting Eqs. (43)-(45) into Eq. (42), we have

$$T=\frac{1}{2}(m_1+m_2+m_B)(\dot{q}_{a1}^R)^2+(\frac{m_2l_2}{2}+m_2r)\dot{q}_{a1}^R\dot{q}_{a2}^R\cos q_{a2}^R$$
$$+(\frac{1}{2}m_2r^2+\frac{1}{2}m_2rl_2+\frac{1}{6}m_2l_2^2+\frac{1}{2}J_B)(\dot{q}_{a2}^R)^2 \quad (46)$$

Select the origin at O, the potential energy of the system such as

$$\Pi=(m_1+m_2+m_B)gq_{a1}^R+m_2g(\frac{l_2}{2}+r)\sin q_{a2}^R\,. \quad (47)$$

Expressions for generalizing forces have the following form

$$Q_{q_{a1}^R}^* = F^R-M_{d1}\quad,\quad Q_{q_{a2}^R}^*=\tau^R-M_{d2} \quad (48)$$

Using the Lagrange equations, the differential equations of the manipulator are determined as follows

$$F_{d1}+(m_1+m_2+m_B)\ddot{q}_{a1}^R+(\frac{m_2l_2}{2}+m_2r)\ddot{q}_{a2}^R\cos q_{a2}^R$$
$$-(\frac{m_2l_2}{2}+m_2r)(\dot{q}_{a2}^R)^2\sin q_{a2}^R=-(m_1+m_2+m_B)g+F^R \quad (49)$$

$$M_{d2}+(\frac{1}{2}m_2l_2+m_2r)\ddot{q}_{a1}^R\cos q_{a2}^R+(m_2r^2+m_2rl_2$$
$$+\frac{1}{3}m_2l_2^2+J_B)\ddot{q}_{a2}^R=-m_2g(\frac{l_2}{2}+r)\cos q_{a2}^R+\tau^R\,. \quad (50)$$

We assume that the motion rule of the drive has the following form

$$q_{a1}^R=0.025\cos(\pi t-\frac{\pi}{2})\quad(m) \quad (51)$$

$$q_{a2}^R=\frac{\pi}{4}\cos(\pi t-\frac{\pi}{2})\quad(rad) \quad (52)$$

By differentiating Eqs. (51) and (52) and then substituting the obtained results into Eqs. (49) and (50) we have the force $F^R$ and the torque $\tau^R$. Then we obtain the fundamental motion of the manipulator with virtual rigid links

$$\mathbf{s}^R(t)=\begin{bmatrix}\mathbf{q}_a^R(t)\\ \mathbf{q}_e^R(t)\end{bmatrix}=\begin{bmatrix}\mathbf{q}_a^R(t)\\ \mathbf{0}\end{bmatrix}. \quad (53)$$

and the force and the torque

$$\boldsymbol{\tau}^R=[F^R\ \ \tau^R]^T \quad (54)$$

### 3.3. Linearization of the motion equations about the fundamental motion

In this section, we focus on the linearization of motion equations of flexible manipulator based on inverse dynamics of the virtual rigid motion of flexible link. The differential equations of motion of the two-link rigid-flexible manipulator T-R can be expressed in the compact matrix form [4, 5]

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}}+\mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}}+\mathbf{g}(\mathbf{q})=\boldsymbol{\tau}(t) \quad (55)$$

where $\mathbf{q},\dot{\mathbf{q}}$ and $\ddot{\mathbf{q}}$ are the generalized coordinates, velocities, and accelerations, respectively. If we choose N=1, the generalized coordinates, velocities, and accelerations of the rigid-flexible two-link manipulator T-R have the following form

$$\mathbf{q}=[q_{a1},q_{a2},q_e]^T \quad (56)$$

$$\boldsymbol{\tau}(t)=[\tau_{a1}(t),\tau_{a2}(t),\tau_e(t)]^T=[\tau_{a1}(t),\tau_{a2}(t),0]^T \quad (57)$$

Let $\Delta q_{a1}$, $\Delta q_{a2}$ and $\Delta q_e$ are the difference between the real motion $\mathbf{q}(t)$ and the fundamental motion $\mathbf{q}^R(t)$, we have

$$y_1(t)=q_{a1}(t)-q_{a1}^R(t),\ \ y_2(t)=q_{a2}(t)-q_{a2}^R(t),\quad y_3(t)=q_e(t) \quad (58)$$

where $y_1$, $y_2$ and $y_3$ are called the perturbed motion.

Similarly, it follows that

$$\boldsymbol{\tau}(t)=[F(t),\tau(t),\tau_e(t)]^T=[F(t),\tau(t),0]^T \quad (59)$$

The elements $q_{a1}^R(t),q_{a2}^R(t)$ are given in Eqs.(51) and (52).

By substituting Eqs. (58) and (59) into Eq. (55) and using Taylor series expansion [9, 10] around fundamental motion, then neglecting nonlinear terms, we obtain the system of linear differential equations with time-varying coefficients for the two-link rigid-flexible manipulator

T-R as follows

$$\mathbf{M}_L(t)\ddot{\mathbf{y}} + \mathbf{C}_L(t)\dot{\mathbf{y}} + \mathbf{K}_L(t)\mathbf{y} = \mathbf{h}_L(t) \quad . \tag{60}$$

The matrices $\mathbf{M}_L(t), \mathbf{C}_L(t), \mathbf{K}_L(t)$ and vector $\mathbf{h}_L(t)$ of the linear differential equations according to Eq. (55) has elements given by:

$$\begin{aligned} &m_{11} = m_1 + m_2 + m_B, m_{22} = J_B + m_2(r^2 + l_2 + \frac{l_2^2}{3}) \\ &m_{12} = m_{21} = \frac{m_2(l_2 + 2r)}{2}\cos q_{a2}^R, m_{33} = \mu n_{11}, \\ &m_{13} = m_{31} = \mu C_1 \cos q_{a2}^R, m_{23} = m_{32} = \mu r C_1 + \mu D_1 \end{aligned} \tag{61}$$

$$\begin{aligned} &c_{11} = \alpha_1, c_{22} = \alpha_2, c_{12} = -(m_2 l_2 + 2m_2 r)\dot{q}_{a2}^R \sin q_{a2}^R \\ &c_{21} = c_{31} = c_{32} = c_{33} = 0, c_{23} = -2\mu C_1 \dot{q}_{a2}^R \sin q_{a2}^R \end{aligned} \tag{62}$$

$$\begin{aligned} &k_{11} = k_{21} = k_{31} = 0, k_{33} = EIh_{11} - \mu(\dot{q}_{a2}^R)^2 n_{11}, \\ &k_{12} = -m_2(\frac{l_2}{2} + r)[\ddot{q}_{a2}^R \sin q_{a2}^R + (\dot{q}_{a2}^R)^2 \cos q_{a2}^R], \\ &k_{13} = -\mu C_1[\ddot{q}_{a2}^R \sin q_{a2}^R + (\dot{q}_{a2}^R)^2 \cos q_{a2}^R], \\ &k_{22} = -(\frac{1}{2}m_2 l_2 + m_2 r)(\ddot{q}_{a1}^R + g)\sin q_{a2}^R, \\ &k_{23} = k_{32} = -\mu C_1(\ddot{q}_{a1}^R + g)\sin q_{a2}^R \end{aligned} \tag{63}$$

$$\begin{aligned} &h_1 = h_2 = 0, \\ &h_3 = -\mu g C_1 \cos q_{a2}^R - \mu C_1 \ddot{q}_{a1}^R \cos q_{a2}^R - \ \mu r C_1 + \mu D_1 \ \ \ddot{q}_{a2}^R \end{aligned} \tag{64}$$

$$\mathbf{h}_L(t) = \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix} + \Delta\boldsymbol{\tau} \ . \tag{65}$$

## 4. Linearization of the motion equations of a rigid-flexible two-link manipulator R-R with a moving payload mass

### 4.1. Derivation of equations of motion for flexible manipulators R-R using the floating frame of the reference approach

Let us consider now the vertical-planar motion of a two-link rigid-flexible manipulator R-R shown in Fig. 2.

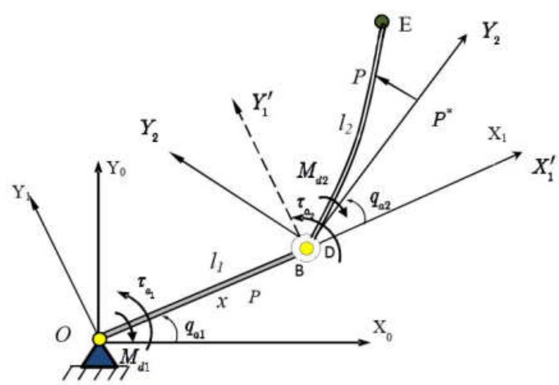


**Fig 2.** Two-link rigid-flexible manipulator R-R

Rigid link OB (link 1) is assumed to be uniform. Flexible link DE (link 2) is clamped to the rigid moving base and assumed to be thin, uniform, and satisfies the Euler-Bernoulli beam assumptions of small shear and rotary inertia effects. Both links are connected by disc B which moves on the plane. The payload mass at the free end of link 2 can move without friction relative to the link using a force actuator. Using the floating frame of reference approach [6], the motion equations of the two-link rigid-flexible manipulator are derived.

Using the floating frame of reference approach [6], the motion equations of the two-link rigid-flexible manipulator R-R (Fig. 2) are derived as described below. For kinematics analysis, the position of point P on the flexible beam is given as

$$\begin{aligned} x_P &= l_1\cos(q_{a1}) + (r + x)\cos(q_{a1} + q_{a2}) \\ &- w(x,t)\sin(q_{a1} + q_{a2}) \end{aligned} \tag{66}$$

$$\begin{aligned} y_P &= l_1\sin(q_{a1}) + (r + x)\sin(q_{a1} + q_{a2}) \\ &+ w(x,t)\cos(q_{a1} + q_{a2}) \end{aligned} \tag{67}$$

Differentiation of Eqs. (66) and (67) yields

$$\begin{aligned} v_P^2 &= l_1^2\dot{q}_{a1}^2 + [(r + x)^2 + w^2](\dot{q}_{a1} + \dot{q}_{a2})^2 \\ &+ \dot{w}^2 + 2l_1(r + x)\dot{q}_{a1}(\dot{q}_{a1} + \dot{q}_{a2})\cos(q_{a2}) \\ &+ 2(r + x)\dot{w}(\dot{q}_{a1} + \dot{q}_{a2}) + 2l_1\dot{q}_{a1}\dot{w}\cos(q_{a2}) \\ &- 2l_1 w\dot{q}_{a1}(\dot{q}_{a1} + \dot{q}_{a2})\sin(q_{a2}) \end{aligned} \tag{68}$$

It follows that

$$\begin{aligned} v_E^2 &= l_1^2\dot{q}_{a1}^2 + [(r + l_2)^2 + w_E^2](\dot{q}_{a1} + \dot{q}_{a2})^2 \\ &+ \dot{w}_E^2 + 2l_1(r + l_2)\dot{q}_{a1}(\dot{q}_{a1} + \dot{q}_{a2})\cos(q_{a2}) \\ &+ 2(r + l_2)\dot{w}_E(\dot{q}_{a1} + \dot{q}_{a2}) + 2l_1\dot{q}_{a1}\dot{w}_E\cos(q_{a2}) \\ &- 2l_1 w_E\dot{q}_{a1}(\dot{q}_{a1} + \dot{q}_{a2})\sin(q_{a2}) \end{aligned} \tag{69}$$

The Euler-Bernoulli beam theory and Ritz-Galerkin method are then applied to the flexible manipulator with assuming that the deformation in the longitudinal direction is negligibly small. Let the transverse deformation of the beam be written as

$$w(x,t) = \sum_{i=1}^{N} X_i(x)q_{ei}(t), \ w_E = \sum_{i=1}^{N} X_i(l_2)q_{ei}(t) \ , \tag{70}$$

where $q_{ei}(t)$ denotes the modal coordinates of transverse deformation, $X_i(x)$ the mode shapes of transverse deformation of a clamped- free beam. The mode shapes are given as [7]

$$\begin{aligned} X_i(x) &= \cos(\beta_i x) - \cosh(\beta_i x) \\ &+ \frac{\cos\beta_i l + \cosh\beta_i l}{\sin\beta_i l + \sinh\beta_i l}(-\sin\beta_i x + \sinh\beta_i x) \end{aligned} \tag{71}$$

The kinetic energy of the flexible manipulator shown in Fig. 2 is given by

$$T = T_{OB} + T_B + T_{DE} + T_E$$

$$T = \frac{1}{2}J_1\dot{q}_{a1}^2 + \frac{1}{2}m_B l_1^2\dot{q}_{a1}^2 + \frac{1}{2}J_B(\dot{q}_{a1} + \dot{q}_{a2})^2 + \frac{1}{2}m_E v_E^2 + \frac{1}{2}\int_0^{l_2}\rho A v_P^2 dx \tag{72}$$

where $J_1$ is the mass moment of inertia of link 1 (including the motor) with respect to the point O, $m_B$ and $J_B$ are the mass and the mass moment of inertia of disc B respectively, $m_E$ is the mass of point E, $\rho A$ is the mass per unit length of the beam. Substituting Eqs. (68 - 71) into Eq. (72), the kinetic energy of the manipulator can be written as

$$\begin{aligned}
T = &\left[\frac{1}{2}J_1 + \frac{1}{2}J_B + \frac{1}{2}(m_2 + m_B)l_1^2 + \frac{1}{2}m_2 r^2 + \frac{1}{2}m_2 r l_2\right.\\
&+\frac{1}{2}m_E l_1^2 + [(m_2 r l_1 + \frac{1}{2}m_2 l_1 l_2) + m_E l_1(r + l_2)]\cos(q_{a2})\\
&+\frac{1}{6}m_2 l_2^2 + \frac{1}{2}m_E(r + l_2)^2 - m_E l_1\sin(q_{a2})\sum_{i=1}^{N}X_i(l_2)q_{ei}\\
&\left.+\frac{1}{2}m_E\sum_{i=1}^{N}\sum_{j=1}^{N}X_i(l_2)X_j(l_2)q_{ei}q_{ej}\right]\dot{q}_{a1}^2 + \left[\frac{1}{2}J_B + \frac{1}{2}m_2 r^2\right.\\
&+\frac{1}{2}m_2 r l_2 + \frac{1}{2}m_E\sum_{i=1}^{N}\sum_{j=1}^{N}X_i(l_2)X_j(l_2)q_{ei}q_{ej} + \frac{1}{6}m_2 l_2^2\\
&\left.+\frac{1}{2}m_E(r + l_2)^2\right]\dot{q}_{a2}^2 + \left[J_B + m_2 r^2 + m_2 r l_2 + \frac{1}{3}m_2 l_2^2\right.\\
&+(m_2 r l_1 + \frac{1}{2}m_2 l_1 l_2)\cos(q_{a2}) + m_E l_1(r + l_2)\cos(q_{a2})\\
&+m_E\sum_{i=1}^{N}\sum_{j=1}^{N}X_i(l_2)X_j(l_2)q_{ei}q_{ej} + m_E(r + l_2)^2\\
&\left.-m_E l_1\sin(q_{a2})\sum_{i=1}^{N}X_i(l_2)q_{ei}\right]\dot{q}_{a1}\dot{q}_{a2}\\
&+\rho A l_1\dot{q}_{a1}\cos(q_{a2})\sum_{i=1}^{N}C_i\dot{q}_{ei}\\
&+m_E l_1\dot{q}_{a1}\cos(q_{a2})\sum_{i=1}^{N}X_i(l_2)\dot{q}_{ei}\\
&+\frac{1}{2}m_E\sum_{i=1}^{N}\sum_{j=1}^{N}X_i(l_2)X_j(l_2)\dot{q}_{ei}\dot{q}_{ej}\\
&+m_E(r + l_2)(\dot{q}_{a1} + \dot{q}_{a2})\sum_{i=1}^{N}X_i(l_2)\dot{q}_{ei}\\
&+\frac{1}{2}\rho A(\dot{q}_{a1} + \dot{q}_{a2})^2\sum_{i=1}^{N}\sum_{j=1}^{N}n_{ij}q_{ei}q_{ej}\\
&-\rho A l_1\dot{q}_{a1}(\dot{q}_{a1} + \dot{q}_{a2})\sin(q_{a2})\sum_{i=1}^{N}C_i q_{ei}
\end{aligned} \tag{73}$$

where

$$C_i = \int_0^{l_2}X_i dx\ ;\ \ D_i = \int_0^{l_2}xX_i dx\ ;\ \ n_{ij} = \int_0^{l_2}X_i X_j dx \tag{74}$$

The expression of the bending deformation of the flexible link has the following form [8]

$$\Pi_e = \frac{1}{2}EI\int_0^{l_2}\left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx\,, \tag{75}$$

where $EI$ is the beam flexural rigidity. By substituting Eqs. (70), (71) into Eq. (75) and adding the gravitational potential energy, we obtain

$$\begin{aligned}
\Pi = &[m_1 a_1 + m_E l_1 + m_2 l_1 + m_B l_1]g\sin(q_{a1})\\
&+[m_E(r + l_2) + m_2 r + \frac{1}{2}m_2 l_2]g\sin(q_{a1} + q_{a2})\\
&+\frac{1}{2}EI\sum_{i=1}^{N}\sum_{j=1}^{N}k_{ij}^* q_{ei}q_{ej} + \rho A g\cos(q_{a1} + q_{a2})\sum_{i=1}^{N}C_i q_{ei}\\
&+m_E g\cos(q_{a1} + q_{a2})\sum_{i=1}^{N}X_i(l_2)q_{ei}
\end{aligned} \tag{76}$$

where

$$k_{ij}^* = \int_0^{l_2}X_i'' X_j'' dx \tag{77}$$

Lagrange equations have the following form [4, 5]

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = -\frac{\partial \Pi}{\partial q_j} + Q_j^*\,,\ \text{j=1,2,…,n}\,, \tag{78}$$

where $q_j$ are the generalized coordinates which include rigid body coordinate $q_a$ as well as elastic modal $q_{ei}$, and $Q_j^*$ are generalized forces. In this paper $Q_j^* = \tau_{aj} + M_{dj}$, in which $M_{dj}$ is damping force which has the following form

$$M_{d1} = \alpha_1\dot{q}_{a1},\quad M_{d2} = \alpha_2\dot{q}_{a2}\,. \tag{79}$$

Substitution of Eqs. (71), (74) and (77) into Eq. (76) yields the equations of motion of the considered system as follows

$$\begin{aligned}
&\left[J_1 + J_B + (m_2 + m_B)l_1^2 + m_2 r^2 + m_2 r l_2 + \frac{1}{3}m_2 l_2^2 + m_E l_1^2\right.\\
&+(2m_2 r l_1 + m_2 l_1 l_2)\cos(q_{a2}) + \rho A\sum_{i=1}^{N}\sum_{j=1}^{N}n_{ij}q_{ei}q_{ej}\\
&-2m_E l_1\sin(q_{a2})\sum_{i=1}^{N}X_i(l_2)q_{ei} + 2m_E l_1(r + l_2)\cos(q_{a2})\\
&+m_E\sum_{i=1}^{N}\sum_{j=1}^{N}X_i(l_2)X_j(l_2)q_{ei}q_{ej} - 2\rho A l_1\sin(q_{a2})\sum_{i=1}^{N}C_i q_{ei}\\
&\left.+m_E(r + l_2)^2\right]\ddot{q}_{a1} + \left[J_B + m_2 r^2 + m_2 r l_2 + \frac{1}{3}m_2 l_2^2\right.\\
&+[m_2 r l_1 + \frac{1}{2}m_2 l_1 l_2 + m_E l_1(r + l_2) + m_E(r + l_2)^2]\cos(q_{a2})\\
&+\rho A\sum_{i=1}^{N_1}\sum_{j=1}^{N_1}n_{ij}q_{ei}q_{ej} + m_E\sum_{i=1}^{N}\sum_{j=1}^{N}X_i(l_2)X_j(l_2)q_{ei}q_{ej}\\
&\left.-m_E l_1\sin(q_{a2})\sum_{i=1}^{N}X_i(l_2)q_{ei} - \rho A l_1\sin(q_{a2})\sum_{i=1}^{N}C_i q_{ei}\right]\ddot{q}_{a2} +
\end{aligned}$$

$$
\begin{aligned}
&+\Big[[m_E(r+l_2)+m_E l_1\cos(q_{a2})]\sum_{i=1}^{N}X_i(l_2)+\rho Ar\sum_{i=1}^{N}C_i \\
&+\rho A\sum_{i=1}^{N}D_i+\rho A l_1\cos q_{a2}\sum_{i=1}^{N}C_i\Big]\ddot{q}_{ei}-\big[m_2 l_1(2r+l_2)\sin q_{a2} \\
&+[2m_E l_1\sum_{i=1}^{N}X_i(l_2)q_{ei}+2\rho A l_1]\cos q_{a2}\sum_{i=1}^{N}C_i q_{ei}+ \\
&+2m_E l_1(r+l_2)\sin q_{a2}\big]\dot{q}_{a1}\dot{q}_{a2}-\Big[(m_2 r l_1+\frac{1}{2}m_2 l_1 l_2)\sin q_{a2}+ \\
&+m_E l_1(r+l_2)\sin q_{a2}+m_E l_1\cos(q_{a2})\sum_{i=1}^{N}X_i(l_2)q_{ei} \\
&+\rho A l_1\cos q_{a2}\sum_{i=1}^{N}C_i q_{ei}]\dot{q}_{a2}^2+2\Big[m_E\sum_{i=1}^{N}\sum_{j=1}^{N}X_i(l_2)X_j(l_2)q_{ej} \\
&+\rho A\sum_{i=1}^{N_1}\sum_{j=1}^{N_1}n_{ij}q_{ej}-l_1[m_E\sum_{i=1}^{N}X_i(l_2)+\rho A]\sin q_{a2}\sum_{i=1}^{N}C_i\Big]\dot{q}_{a1}\dot{q}_{ei} \\
&+2\Big[m_E\sum_{i=1}^{N}\sum_{j=1}^{N}X_i(l_2)X_j(l_2)q_{ej}-m_E l_1\sin(q_{a2})\sum_{i=1}^{N}X_i(l_2) \\
&+\rho A\sum_{i=1}^{N_1}\sum_{j=1}^{N_1}n_{ij}q_{ej}-\rho A l_1\sin(q_{a2})\sum_{i=1}^{N}C_i\Big]\dot{q}_{a2}\dot{q}_{ei} \\
&=\tau_{a1}-[m_1 a_1+m_E l_1++(m_2+m_B)l_1]g\cos(q_{a1}) \\
&-[m_E(r+l_2)+m_2 r+\frac{1}{2}m_2 l_2]g\cos(q_{a1}+q_{a2})-M_{d1} \\
&+[\rho Ag\sum_{i=1}^{N}C_i q_{ei}+m_E g\sum_{i=1}^{N}X_i(l_2)q_{ei}]\sin(q_{a1}+q_{a2})
\end{aligned}
\tag{80}
$$

$$
\begin{aligned}
&\Big[J_B+m_2 r^2+m_2 r l_2+\frac{1}{3}m_2 l_2^2+m_E\sum_{i=1}^{N}\sum_{j=1}^{N}X_i(l_2)X_j(l_2)q_{ei}q_{ej} \\
&+[m_2 r l_1+\frac{1}{2}m_2 l_1 l_2)+m_E l_1(r+l_2)]\cos q_{a2}+m_E(r+l_2)^2 \\
&-m_E l_1\sin q_{a2}\sum_{i=1}^{N}X_i(l_2)q_{ei}+\rho A\sum_{i=1}^{N_1}\sum_{j=1}^{N_1}n_{ij}q_{ei}q_{ej} \\
&-\rho A l_1\sin q_{a2}\sum_{i=1}^{N}C_i q_{ei}\Big]\ddot{q}_{a1}+\Big[J_B+m_2 r^2+m_2 r l_2+\frac{1}{3}m_2 l_2^2 \\
&+\rho A\sum_{i=1}^{N}\sum_{j=1}^{N}n_{ij}q_{ei}q_{ej}+m_E\sum_{i=1}^{N}\sum_{j=1}^{N}X_i(l_2)X_j(l_2)q_{ei}q_{ej} \\
&+m_E(r+l_2)^2\Big]\ddot{q}_{a2}+\Big[m_E(r+l_2)\sum_{i=1}^{N}X_i(l_2)+\rho Ar\sum_{i=1}^{N}C_i \\
&+\rho A\sum_{i=1}^{N}D_i\Big]\ddot{q}_{ei}+\Big[m_2 l_1[r+\frac{1}{2}l_2]\sin q_{a2}+m_E l_1(r+l_2)\sin q_{a2} \\
&+m_E l_1\cos q_{a2}\sum_{i=1}^{N}X_i(l_2)q_{ei}+\rho A l_1\cos(q_{a2})\sum_{i=1}^{N}C_i q_{ei}\Big]\dot{q}_{a1}^2 \\
&+2\Big[m_E\sum_{i=1}^{N}\sum_{j=1}^{N}X_i(l_2)X_j(l_2)q_{ej}+\rho A\sum_{i=1}^{N_1}\sum_{j=1}^{N_1}n_{ij}q_{ej}\Big]\dot{q}_{a1}\dot{q}_{ei} \\
&+2\Big[\rho A\sum_{i=1}^{N_1}\sum_{j=1}^{N_1}n_{ij}q_{ej}+m_E\sum_{i=1}^{N}\sum_{j=1}^{N}X_i(l_2)X_j(l_2)q_{ej}\Big]\dot{q}_{a2}\dot{q}_{ei} \\
&=\tau_{a2}-[m_E(r+l_2)+m_2 r+\frac{1}{2}m_2 l_2]g\cos(q_{a1}+q_{a2}) \\
&+[\rho Ag\sum_{i=1}^{N}C_i q_{ei}+m_E g\sum_{i=1}^{N}X_i(l_2)q_{ei}]\sin(q_{a1}+q_{a2})-M_{d2}
\end{aligned}
\tag{81}
$$

$$
\begin{aligned}
&[m_E(r+l_2)X_i(l_2)+(m_E l_1 X_i(l_2)+\rho A l_1 C_i)\cos q_{a2}+\rho A D_i \\
&+\rho Ar C_i]\ddot{q}_{a1}+[m_E(r+l_2)X_i(l_2)+\rho Ar C_i+\rho A D_i]\ddot{q}_{a2} \\
&+[m_E X_i(l_2)\sum_{j=1}^{N}X_j(l_2)+\rho A\sum_{j=1}^{N}n_{ij}]\ddot{q}_{ej}+[m_E l_1 X_i(l_2)\sin q_{a2} \\
&-m_E X_i(l_2)\sum_{j=1}^{N}X_j(l_2)q_{ej}-\rho A\sum_{j=1}^{N_1}n_{ij}q_{ej}+\rho A l_1 C_i\sin q_{a2}]\dot{q}_{a1}^2 \\
&+[-m_E X_i(l_2)\sum_{j=1}^{N}X_j(l_2)q_{ej}-\rho A\sum_{j=1}^{N}n_{ij}q_{ej}]\dot{q}_{a2}^2 \\
&+[-2m_E X_i(l_2)\sum_{j=1}^{N}X_j(l_2)q_{ej}-2\rho A\sum_{j=1}^{N}n_{ij}q_{ej}]\dot{q}_{a1}\dot{q}_{a2} \\
&=-m_E g X_i(l_2)\cos(q_{a1}+q_{a2})-\rho Ag C_i\cos(q_{a1}+q_{a2}) \\
&-EI\sum_{j=1}^{N}k_{ij}^{*}q_{ej},\quad i=1,2,\ldots N
\end{aligned}
\tag{82}
$$

If we choose N = 1, the differential equations of motion of the rigid-flexible two-link manipulator have the following form

$$
\begin{aligned}
&\big[J_1+J_B+m_2 r l_2+m_E l_1^2+\frac{1}{3}m_2 l_2^2+(m_E X_1^2(l_2)+\rho A n_{11})q_{e1}^2 \\
&+m_E(r+l_2)^2+[2m_2 r l_1+m_2 l_1 l_2+2m_E l_1(r+l_2)]\cos q_{a2} \\
&+(m_2+m_B)l_1^2+m_2 r^2-2l_1 q_{e1}[\rho A C_1+m_E X_1(l_2)]\sin q_{a2}\big]\ddot{q}_{a1} \\
&+\Big[J_B+m_2 r^2+m_2 r l_2+\frac{1}{3}m_2 l_2^2+(m_2 r l_1+\frac{1}{2}m_2 l_1 l_2)\cos q_{a2} \\
&+m_E(r+l_2)^2+m_E X_1^2(l_2)q_{e1}^2+m_E l_1(r+l_2)\cos q_{a2} \\
&-[m_E l_1 X_1(l_2)q_{e1}+\rho A l_1 C_1 q_{e1}]\sin q_{a2}+\rho A n_{11}q_{e1}^2\big]\ddot{q}_{a2} \\
&+\big[m_E(r+l_2)X_1(l_2)+[m_E l_1 X_1(l_2)+\rho A l_1 C_1]\cos q_{a2}+\rho A D_1 \\
&+\rho Ar C_1\big]\ddot{q}_{e1}-\big[[2m_2 r l_1+m_2 l_1 l_2+2m_E l_1(r+l_2)]\sin q_{a2} \\
&+[2m_E l_1 X_1(l_2)q_{e1}+2\rho A l_1 C_1 q_{e1}]\cos q_{a2}\big]\dot{q}_{a1}\dot{q}_{a2} \\
&-\big[[m_2 r l_1+\frac{1}{2}m_2 l_1 l_2+m_E l_1(r+l_2)]\sin q_{a2}+[m_E l_1 X_1(l_2)q_{e1} \\
&+\rho A l_1 C_1 q_{e1}]\cos q_{a2}\big]\dot{q}_{a2}^2+2\big[m_E X_1^2(l_2)q_{e1}+\rho A n_{11}q_{e1} \\
&-[m_E l_1 X_1(l_2)+\rho A l_1 C_1\sin q_{a2}]\big]\dot{q}_{a1}\dot{q}_{e1}+2\big[m_E X_1^2(l_2)q_{e1} \\
&-[m_E l_1 X_1(l_2)+\rho A l_1 C_1]\sin q_{a2}+\rho A n_{11}q_{e1}\big]\dot{q}_{a2}\dot{q}_{e1} \\
&=\tau_{a1}-[m_1 a_1+m_E l_1+(m_2+m_B)l_1]g\cos q_{a1} \\
&+[m_E g X_1(l_2)q_{e1}+\rho Ag C_1 q_{e1}]\sin(q_{a1}+q_{a2}) \\
&-[m_E(r+l_2)+m_2 r+\frac{1}{2}m_2 l_2]g\cos(q_{a1}+q_{a2})-\alpha_1\dot{q}_{a1}
\end{aligned}
\tag{83}
$$

$$
\begin{aligned}
&\Big[J_B + m_2 r^2 + m_2 r l_2 + \frac{1}{3} m_2 l_2^2 + m_E X_1^2(l_2) q_{e1}^2 + \rho A n_{11} q_{e1}^2 \\
&+[m_E l_1 (r+l_2) + m_2 r l_1 + \frac{1}{2} m_2 l_1 l_2\,]\cos q_{a2} + m_E (r+l_2)^2 \\
&-[m_E l_1 X_1(l_2) q_{e1} + \rho A l_1 C_1 q_{e1}]\sin q_{a2}\Big]\ddot{q}_{a1} + \Big[m_2(r^2 + r l_2) \\
&+J_B + \frac{1}{3} m_2 l_2^2 + m_E (r+l_2)^2 + m_E X_1^2(l_2) q_{e1}^2 + \rho A n_{11} q_{e1}^2\Big]\ddot{q}_{a2} \\
&+[m_E (r+l_2)\, X_1(l_2) + \rho A r C_1 + \rho A D_1]\ddot{q}_{e1} + \Big[(m_2 r l_1 \\
&+\frac{1}{2} m_2 l_1 l_2)\sin q_{a2} + m_E l_1 (r+l_2)\sin q_{a2} + (m_E l_1 X_1(l_2) q_{e1} \\
&+\rho A l_1 C_1 q_{e1})\cos q_{a2}\Big]\dot{q}_{a1}^2 + [2 m_E X_1^2(l_2) q_{e1} + 2\rho A n_{11} q_{e1}]\dot{q}_{a1}\dot{q}_{e1} \\
&+[2 m_E X_1^2(l_2) q_{e1} + 2\rho A n_{11} q_{e1}]\dot{q}_{a2}\dot{q}_{e1} \\
&= \tau_{a2} - [m_E (r+l_2) + m_2 r + \frac{1}{2} m_2 l_2]g\cos(q_{a1}+q_{a2}) \\
&+[m_E g X_1(l_2) q_{e1} + \rho A g C_1 q_{e1}]\sin(q_{a1}+q_{a2}) - \alpha_2 \dot{q}_{a2}
\end{aligned}
\tag{84}
$$

$$
\begin{aligned}
&\Big[m_E (r+l_2) X_1(l_2) + [m_E l_1 X_1(l_2) + \rho A l_1 C_1]\cos q_{a2} + \rho A D_1 \\
&+\rho A r C_1\Big]\ddot{q}_{a1} + [m_E (r+l_2)\, X_1(l_2) + \rho A r C_1 + \rho A D_1]\ddot{q}_{a2} \\
&+[m_E X_1^2(l_2) + \rho A n_{11}]\ddot{q}_{e1} + \Big[-m_E X_1^2(l_2) q_{e1} - \rho A m_{11} q_{e1} \\
&+(m_E l_1 X_1(l_2) + \rho A l_1 C_1)\sin q_{a2}\Big]\dot{q}_{a1}^2 + [-m_E X_1^2(l_2) q_{e1} \\
&-\rho A n_{11} q_{e1}]\dot{q}_{a2}^2 + [-2 m_E X_1^2(l_2) q_{e1} - 2\rho A n_{11} q_{e1}]\dot{q}_{a1}\dot{q}_{a2} \\
&= -[m_E g X_1(l_2) - \rho A g C_1]\cos(q_{a1}+q_{a2}) - E I k_{11}^* q_{e1}
\end{aligned}
\tag{85}
$$

### 4.2. Fundamental motion of the flexible manipulator

We consider the case of the manipulator shown in Fig. 2, that consists of the two rigid links, the fundamental motion of the corresponding flexible manipulator is the virtual rigid link motion of the link DE. From the virtual rigid link motion, the position of the point E is given as

$$
\begin{aligned}
x_E^R &= l_1\cos(q_{a1}^R) + \big(r+l_2\big)\cos(q_{a1}^R + q_{a1}^R) \\
y_E^R &= l_1\sin(q_{a1}^R) + \big(r+l_2\big)\sin(q_{a1}^R + q_{a1}^R)
\end{aligned}
\tag{86}
$$

The kinetic energy of the system is given by

$$T = T_{OB} + T_B + T_{DE} + T_E \tag{87}$$

where

$$T_{OB} = \frac{1}{2} J_1 (\dot{q}_{a1}^R)^2,\; T_B = \frac{1}{2} m_B l_1^2 (\dot{q}_{a1}^R)^2 + \frac{1}{2} J_B (\dot{q}_{a1}^R + \dot{q}_{a1}^R)^2 \tag{88}$$

$$
\begin{aligned}
T_E &= \frac{1}{2} m_E v_E^2 = \frac{1}{2} m_E [l_1^2 (\dot{q}_{a1}^R)^2 + (r+l_2)^2 (\dot{q}_{a1}^R + \dot{q}_{a2}^R)^2 \\
&\quad + 2 l_1 r + l_2)\dot{q}_{a1}(\dot{q}_{a1}^R + \dot{q}_{a2}^R)\cos q_{a2}^R]
\end{aligned}
\tag{89}
$$

$$
\begin{aligned}
T_{DE} &= m_2[\frac{l_1^2}{2} + \frac{r^2}{2} + \frac{r l_2}{2} + \frac{l_2^2}{6} + l_1(r + \frac{l_2}{2})\cos q_{a2}^R](\dot{q}_{a1}^R)^2 \\
&\quad + m_2[r^2 + r l_2 + \frac{l_2^2}{3} + l_1(r + \frac{l_2}{2})\cos q_{a2}^R]\dot{q}_{a1}^R \dot{q}_{a2}^R \\
&\quad + (\frac{1}{2} m_2 r^2 + \frac{1}{2} m_2 r l_2 + \frac{m_2 l_2^2}{6})(\dot{q}_{a2}^R)^2
\end{aligned}
\tag{90}
$$

From Eqs. (88), (89) and (90), the kinetic energy of the system has the following form

$$
\begin{aligned}
T &= \frac{1}{2}\Big[J_1 + J_B + m_2 l_1^2 + m_B l_1^2 + m_E l_1^2 + m_2 r^2 + m_2 r l_2 + \frac{m_2 l_2^2}{3} \\
&+ m_E (r+l_2)^2 + [2 m_2 l_1 (r + \frac{l_2}{2}) + 2 m_E l_1 (r+l_2)]\cos q_{a2}^R\Big](\dot{q}_{a1}^R)^2 \\
&+ \frac{1}{2}[J_B + m_2 r^2 + m_2 r l_2 + \frac{m_2 l_2^2}{3} + m_E (r+l_2)^2](\dot{q}_{a2}^R)^2 \\
&+ [J_B + m_2 r^2 + m_2 r l_2 + \frac{m_2 l_2^2}{3} + m_E (r+l_2)^2 \\
&+ m_2 l_1 (r + \frac{l_2}{2})\cos q_{a2}^R + m_E l_1 (r+l_2)\cos q_{a2}^R]\dot{q}_{a1}^R \dot{q}_{a2}^R
\end{aligned}
\tag{91}
$$

The potential energy of the system can be expressed in the form

$$
\begin{aligned}
\Pi &= (m_1 \frac{l_1}{2} + m_2 l_1 + m_B l_1 + m_E l_1) g \sin q_{a_1}^R \\
&+ [m_2 (r + \frac{l_2}{2}) + m_E (r + l_2)] g \sin(q_{a1}^R + q_{a2}^R)
\end{aligned}
\tag{92}
$$

The use of Lagrange equation leads to the differential equations of motion as

$$
\begin{aligned}
\tau_{a1}^R &= \Big[J_1 + J_B + m_2 l_1^2 + m_B l_1^2 + m_E l_1^2 + m_2 r^2 + m_E (r+l_2)^2 \\
&+ m_2 r l_2 + \frac{m_2 l_2^2}{3} + 2[m_2 l_1 (r + \frac{l_2}{2}) + 2 m_E l_1 (r+l_2)\Big]\cos q_{a2}^R]\ddot{q}_{a1}^R \\
&+ \Big[J_B + m_2 (r l_2 + \frac{l_2^2}{3} + r^2) + l_1 [m_2 (r + \frac{l_2}{2}) + m_E (r+l_2)]\cos q_{a2}^R \\
&+ m_E (r+l_2)^2\Big]\ddot{q}_{a2}^R - 2 l_1 [m_2 (r + \frac{l_2}{2}) + m_E (r+l_2)]\dot{q}_{a1}^R \dot{q}_{a2}^R \sin q_{a2}^R \\
&- [m_2 l_1 (r + \frac{l_2}{2}) + m_E l_1 (r+l_2)](\dot{q}_{a2}^R)^2 \sin q_{a2}^R \\
&+ [m_2 (r + \frac{l_2}{2}) + m_E (r+l_2)] g \cos(q_{a1}^R + q_{a2}^R) \\
&+ (\frac{m_1}{2} + m_2 + m_B + m_E) l_1 g \cos q_{a1}^R + M_{d1}^R
\end{aligned}
\tag{93}
$$

$$
\begin{aligned}
\tau_{a2}^R &= \Big[J_B + m_2 r^2 + m_2 r l_2 + l_1 [m_2 (r + \frac{l_2}{2}) + m_E (r+l_2)]\cos q_{a2}^R \\
&+ \frac{m_2 l_2^2}{3} + m_E (r+l_2)^2\Big]\ddot{q}_{a1}^R + \Big[J_B + m_2 r^2 + m_E (r+l_2)^2 + \frac{m_2 l_2^2}{3} \\
&+ m_2 r l_2\Big]\ddot{q}_{a2}^R + [m_2 l_1 (r + \frac{l_2}{2})\sin q_{a2}^R + m_E l_1 (r+l_2)\sin q_{a2}^R]\dot{q}_{a1}^2 \\
&+ [m_2 (r + \frac{l_2}{2}) + m_E (r+l_2)] g \cos(q_{a1}^R + q_{a2}^R) + \mathrm{M}_{d2}^R
\end{aligned}
\tag{94}
$$

We assume that the motion rule of the drive has the following form

$$q_{a1}^R(t) = \frac{\pi}{2} + \frac{\pi}{4}\sin(\Omega t) \tag{95}$$

$$q_{a2}^R(t) = \frac{\pi}{2}\cos(\Omega t) \quad (96)$$

It should be noted that the fundamental motion of the manipulator is described by $\mathbf{q}^R(t)$ and $\boldsymbol{\tau}^R(t)$, where $\mathbf{q}^R(t)$ are the generalized coordinates of the manipulator and $\boldsymbol{\tau}^R(t)$ the torques

$$\mathbf{q}^R(t) = \begin{bmatrix} q_{a1}^R(t) & q_{a2}^R(t) & q_e^R(t) \end{bmatrix}^T = \begin{bmatrix} q_{a1}^R(t) & q_{a2}^R(t) & 0 \end{bmatrix}^T \quad (97)$$

$$\boldsymbol{\tau}^R(t) = \begin{bmatrix} \tau_{a1}^R & \tau_{a2}^R & \tau_e^R \end{bmatrix}^T = \begin{bmatrix} \tau_{a1}^R & \tau_{a2}^R & 0 \end{bmatrix}^T \quad (98)$$

where $q_e^R(t)$ denotes the elastic generalized coordinate and $\tau_e^R(t)$ is the elastic torque of the virtual rigid link. The torques can be determined by differentiating Eqs. (95), (96) and then substituting the results into Eqs. (93) and (94).

**4.3. Linearization of the motion equations about the fundamental motion**

In this section, we focus on the linearization of motion equations of flexible manipulator based on inverse dynamics of the virtual rigid motion of flexible link. The differential equations of motion of the two-link rigid-flexible manipulator R-R can be written in matrix form as follows [1-5]

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) = \boldsymbol{\tau}(t) \quad (99)$$

Where $\mathbf{q}$, $\dot{\mathbf{q}}$ and $\mathbf{q}$ are the generalized coordinates, velocities, and accelerations, respectively. If we choose N=1, the generalized coordinates, velocities, and accelerations of the rigid-flexible two-link manipulator have the following form

$$\mathbf{q} = \begin{bmatrix} q_{a1}, q_{a2}, q_e \end{bmatrix}^T \quad (100)$$

Let $\Delta q_{a1}$, $\Delta q_{a2}$ and $q_e$ are the difference between the real motion $\mathbf{q}(t)$ and the fundamental motion $\mathbf{q}^R(t)$, we have

$$\begin{aligned} &y_1(t) = \Delta q_{a1}(t) = q_{a1}(t) - q_{a1}^R(t), \\ &y_2(t) = \Delta q_{a2}(t) = q_{a2}(t) - q_{a2}^R(t), y_3(t) = q_e(t) \end{aligned} \quad (101)$$

where $y_1, y_2$ and $y_3$ are called the perturbed motion. Similarly, it follows that

$$\boldsymbol{\tau}(t) = \begin{bmatrix} \tau_{a1}(t), \tau_{a2}(t), \tau_e(t) \end{bmatrix}^T = \begin{bmatrix} \tau_{a1}(t), \tau_{a2}(t), 0 \end{bmatrix}^T \quad (102)$$

The elements $q_{a1}^R(t), q_{a2}^R(t)$ are given by Eqs. (92) and (93) By substituting Eqs. (101) into Eq. (99) and using Taylor series expansion [9, 10] around fundamental motion, then neglecting nonlinear terms, we obtain the system of linear differential equations with time-varying coefficients for the two-link rigid - flexible manipulator R-R as follows

$$\mathbf{M}_L(t)\ddot{\mathbf{y}} + \mathbf{C}_L(t)\dot{\mathbf{y}} + \mathbf{K}_L(t)\mathbf{y} = \mathbf{h}_L(t) \quad (103)$$

The matrices $\mathbf{M}_L(t)$, $\mathbf{C}_L(t)$, $\mathbf{K}_L(t)$ and vector $\mathbf{h}_L(t)$ of the linear differential equations according to Eq. (99) have the following forms:

$$\mathbf{M}_L(t) = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}, \mathbf{C}_L(t) = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \quad (104)$$

$$\mathbf{K}_L(t) = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix}, \mathbf{h}_L(t) = \begin{bmatrix} h_1 & h_2 & h_3 \end{bmatrix}^T \quad (105)$$

Elements of the coefficient matrices in Eqs. (104) and (105) have the following forms

$$\begin{aligned} m_{11} &= J_1 + J_B + (m_2 + m_B)l_1^2 + m_2(r^2 + rl_2 + \frac{1}{3}l_2^2) + m_E l_1^2 \\ &+ [2m_2 rl_1 + m_2 l_1 l_2 + 2m_E l_1(r + l_2)]\cos q_{a2}^R + m_E(r + l_2)^2 \end{aligned}$$

$$\begin{aligned} m_{12} = m_{21} &= J_B + m_2 rl_2 + (m_2 rl_1 + \frac{1}{2}m_2 l_1 l_2)\cos q_{a2}^R \\ &+ m_2 r^2 + \frac{1}{3}m_2 l_2^2 + m_E(r + l_2)^2 + m_E l_1(r + l_2)\cos q_{a2}^R \end{aligned}$$

$$\begin{aligned} m_{13} = m_{31} &= m_E(r + l_2)X_1(l_2) + m_E l_1 X_1(l_2)\cos q_{a2}^R \\ &+ \rho ArC_1 + \rho AD_1 + \rho Al_1 C_1 \cos q_{a2}^R \end{aligned}$$

$$m_{22} = J_B + m_2 r^2 + m_2 rl_2 + \frac{1}{3}m_2 l_2^2 + m_E(r + l_2)^2$$

$$m_{23} = m_{32} = m_E(r + l_2)X_1(l_2) + \rho ArC_1 + \rho AD_1$$

$$m_{33} = m_E X_1^2(l_2) + \rho An_{11}$$

$$c_{11} = \alpha_1 - [2m_2 rl_1 + m_2 l_1 l_2 + 2m_E l_1(r + l_2)](\dot{q}_{a2}^R)\sin q_{a2}^R$$

$$c_{12} = -[2m_2 rl_1 + m_2 l_1 l_2 + 2m_E l_1(r + l_2)](\dot{q}_{a1}^R + \dot{q}_{a2}^R)\sin q_{a2}^R$$

$$c_{13} = -[2m_E l_1 X_1(l_2) + 2\rho Al_1 C_1](\dot{q}_{a1}^R + \dot{q}_{a2}^R)\sin q_{a2}^R$$

$$c_{21} = [2(m_2 rl_1 + \frac{1}{2}m_2 l_1 l_2) + 2m_E l_1(r + l_2)]\dot{q}_{a1}^R \sin q_{a2}^R$$

$$c_{22} = \alpha_2; c_{23} = 0; c_{32} = 0; c_{33} = 0 \quad ;$$

$$c_{31} = [2m_E l_1 X_1(l_2) + 2\rho Al_1 C_1]\dot{q}_{a1}^R \sin q_{a2}^R$$

$$\begin{aligned} k_{11} &= -[m_1 a_1 + m_E l_1 + (m_2 + m_B)l_1]g\sin q_{a1}^R \\ &- [m_E(r + l_2) + m_2 r + \frac{1}{2}m_2 l_2]g\sin(q_{a1}^R + q_{a2}^R) \end{aligned}$$

$$\begin{aligned} k_{12} &= [-2m_2 rl_1 - m_2 l_1 l_2 - 2m_E l_1(r + l_2)]\ddot{q}_{a1}^R \sin q_{a2}^R \\ &- [2m_2 rl_1 - m_2 l_1 l_2 - 2m_E l_1(r + l_2)]\dot{q}_{a1}^R \dot{q}_{a2}^R \cos q_{a2}^R \\ &- [m_2 rl_1 + \frac{1}{2}m_2 l_1 l_2 + m_E l_1(r + l_2)](\dot{q}_{a2}^R)^2 \cos q_{a2}^R \end{aligned}$$

$$k_{21} = -[m_E(r + l_2) + m_2 r + \frac{1}{2}m_2 l_2]g\sin(q_{a1}^R + q_{a2}^R)$$

$$k_{13} = [-2\rho A l_1 C_1 - 2m_E l_1 X_1(l_2)]\ddot{q}_{a1}^R \sin q_{a2}^R + [-m_E l_1 X_1(l_2)$$
$$-\rho A l_1 C_1]\ddot{q}_{a2}^R \sin q_{a2}^R - 2l_1[m_E X_1(l_2) + 2\rho A C_1]\dot{q}_{a1}^R \dot{q}_{a2}^R \cos q_{a2}^R$$
$$-[m_E l_1 X_1(l_2)\cos q_{a2}^R + \rho A l_1 C_1 \cos q_{a2}^R](\dot{q}_{a2}^R)^2$$
$$-\rho A g C_1 \sin(q_{a1}^R + q_{a2}^R) - m_E g X_1(l_2)\sin(q_{a1}^R + q_{a2}^R)$$

$$k_{22} = [-m_2 r l_1 - \frac{1}{2} m_2 l_1 l_2 - m_E l_1 (r + l_2)]\ddot{q}_{a1}^R \sin q_{a2}^R$$
$$+[(m_2 r l_1 + \frac{1}{2} m_2 l_1 l_2)\cos q_{a2}^R + m_E l_1 (r + l_2)\cos q_{a2}^R](\dot{q}_{a1}^R)^2$$
$$-[m_E (r + l_2) + m_2 r + \frac{1}{2} m_2 l_2]g\sin(q_{a1}^R + q_{a2}^R)$$

$$k_{23} = [-m_E l_1 X_1(l_2) - \rho A l_1 C_1]\ddot{q}_{a1}^R \sin q_{a_2}^R + [m_E l_1 X_1(l_2) +$$
$$\rho A l_1 C_1](\dot{q}_{a1}^R)^2 \cos q_{a2}^R - [m_E X_1(l_2) + \rho A C_1]g\sin(q_{a1}^R + q_{a2}^R)$$

$$k_{31} = -m_E g X_1(l_2)\sin(q_{a1}^R + q_{a2}^R) - \rho A g C_1 \sin(q_{a1}^R + q_{a2}^R)$$

$$k_{32} = [-m_E l_1 X_1(l_2) - \rho A l_1 C_1]\ddot{q}_{a1}^R \sin q_{a2}^R + [m_E l_1 X_1(l_2) +$$
$$\rho A l_1 C_1](\dot{q}_{a1}^R)^2 \cos q_{a2}^R - [m_E X_1(l_2) + \rho A C_1]g\sin(q_{a1}^R + q_{a2}^R)$$

$$k_{33} = -[m_E X_1^2(l_2) + \rho A n_{11}](\dot{q}_{a1}^R + \dot{q}_{a2}^R)^2 + EIk_{11}^*$$

$$h_1 = \tau_{a1}^R + [2m_2 r l_1 + m_2 l_1 l_2 + 2m_E l_1 (r + l_2)]\dot{q}_{a1}^R \dot{q}_{a2}^R \sin q_{a2}^R$$
$$+[m_2 r l_1 + \frac{1}{2} m_2 l_1 l_2 + m_E l_1 (r + l_2)](\dot{q}_{a2}^R)^2 \sin q_{a2}^R$$
$$-[m_1 a_1 + m_E l_1 + (m_2 + m_B) l_1]g\cos q_{a1}^R - [m_E (r + l_2)$$
$$+m_2 r + \frac{1}{2} m_2 l_2]g\cos(q_{a1}^R + q_{a2}^R) - [J_1 + J_B + (m_2 + m_B) l_1^2$$
$$+m_2 r^2 + m_2 r l_2 + \frac{1}{3} m_2 l_2^2 + m_E l_1^2 + m_E (r + l_2)^2$$
$$+(2m_2 r l_1 + m_2 l_1 l_2)\cos q_{a2}^R + 2m_E l_1 (r + l_2)\cos q_{a2}^R]\ddot{q}_{a1}^R$$
$$-[J_B + m_2 r^2 + m_2 r l_2 + \frac{1}{3} m_2 l_2^2 + (m_2 r l_1 + \frac{1}{2} m_2 l_1 l_2)\cos q_{a2}^R$$
$$+m_E (r + l_2)^2 + m_E l_1 (r + l_2)\cos q_{a2}^R]\ddot{q}_{a2}^R$$

$$h_2 = \tau_{a2}^R + [-m_2 r l_1 - \frac{1}{2} m_2 l_1 l_2 - m_E l_1 (r + l_2)](\dot{q}_{a1}^R)^2 \sin q_{a2}^R$$
$$-[J_B + m_2 r^2 + m_2 r l_2 + \frac{1}{3} m_2 l_2^2 + (m_2 r l_1 + \frac{1}{2} m_2 l_1 l_2)\cos q_{a2}^R$$
$$+m_E (r + l_2)^2 + m_E l_1 (r + l_2)\cos q_{a2}^R]\ddot{q}_{a1}^R - [m_E (r + l_2)$$
$$+m_2 r + \frac{1}{2} m_2 l_2]g\cos(q_{a1}^R + q_{a2}^R) - [J_B + m_2 r^2 + m_2 r l_2$$
$$+\frac{1}{3} m_2 l_2^2 + m_E (r + l_2)^2]\ddot{q}_{a2}^R$$

$$h_3 = [-m_E l_1 X_1(l_2) - \rho A l_1 C_1](\dot{q}_{a1}^R)^2 \sin q_{a2}^R$$
$$-[\rho A r C_1 + \rho A D_1 + m_E (r + l_2) X_1(l_2) + m_E l_1 X_1(l_2)\cos q_{a2}^R$$
$$+\rho A l_1 C_1 \cos q_{a2}^R]\ddot{q}_{a1}^R - [m_E (r + l_2) X_1(l_2) + \rho A (r C_1 + D_1)]\ddot{q}_{a2}^R$$
$$-\rho A g\cos(q_{a1}^R + q_{a2}^R) C_1 - m_E g\cos(q_{a1}^R + q_{a2}^R) X_1(l_2)$$

## 5. Conclusions

In the present paper, the linearization problem of nonlinear motion equations of flexible manipulators in the vicinity of a fundamental motion is addressed. For this, Taylor series expansion of nonlinear motion equations around the desired fundamental motion of the system is included.

The computational algorithm has been implemented and applied to the motion equations of a rigid-flexible two-link manipulator R-R with a moving payload mass and to the motion equations of a rigid-flexible two-link manipulator T-R. After the linearization process, we obtain a system of linear differential equations with coefficient of variation

$$\mathbf{M}_L(t)\ddot{\mathbf{y}} + \mathbf{C}_L(t)\dot{\mathbf{y}} + \mathbf{K}_L(t)\mathbf{y} = \mathbf{h}_L(t)\,.$$

If the matrices M(t), C(t) and K(t) are constant matrices or periodic matrices, we can use analytical methods to find the solution of a linerized system.

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