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Stability Control of Dynamical Systems Described by Linear Differential Equations with Time-Periodic Coefficients

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Abstract. The analysis of dynamical stability is an important problem in the design and control of vibrating structures which are described by a linear differential equation system with time-periodic coefficients. For this kind of system, stable criteria according to the Floquet multipliers is given. In case of an unstable system, a PD controller is added, and its optimal parameters are determined by the Taguchi method.

Keywords: Linearization · Flexible manipulator · Floquet theory · Taguchi method · Stability

1 Introduction

Mathematically, the motion of a multibody system with f degrees of freedom can be described by the following nonlinear differential equation [1–3]

$$\mathbf{M}(\mathbf{q}, t)\ddot{\mathbf{q}} + \mathbf{k}(\dot{\mathbf{q}}, \mathbf{q}, t) = \mathbf{h}(\dot{\mathbf{q}}, \mathbf{q}, t) \tag{1}$$

where $\mathbf{M}(\mathbf{q}, t)$ is the symmetric $f \times f$ inertia matrix, $\mathbf{k}(\dot{\mathbf{q}}, \mathbf{q}, t)$ is the $f \times 1$ vector of the generalized gyroscopic forces, $\mathbf{h}(\dot{\mathbf{q}}, \mathbf{q}, t)$ is the $f \times 1$ vector of the generalized applied forces, and $\mathbf{q}$, $\dot{\mathbf{q}}$, $\ddot{\mathbf{q}}$ are the vectors of generalized position, velocity, acceleration variables, respectively [1]. It is very difficult or impossible to find the analytical solution of Eq. (1). Hence, the numerical methods are the efficient way to solve the problem [1, 2]. The solution of Eq. (1) can be used to simulate the dynamic behavior of multibody systems that undergo large movements.

It is well-known that technical systems work mostly in a neighbourhood of its desired motion which is called the fundamental motion. For instance, the fundamental motion of a driver system is the motion of working components, so that the driver output rotates uniformly, and all components are assumed to be rigid. The fundamental motion of a flexible robotic systems usually described through state variables determined by prescribed motions of the end-effector.

Equation (1) is usually linearized about the fundamental motion to use the linear analysis tools [3–7] for analysing the behavior of the multibody system in the vicinity

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of the fundamental motion. The result of this linearization process leads to a set of linear differential equations with time-varying coefficients. For the case of a flexible manipulator in steady-state motions [5–7], these equations can be written in the matrix form

$$\mathbf{M}_L(t)\ddot{\mathbf{y}} + \mathbf{C}_L(t)\dot{\mathbf{y}} + \mathbf{K}_L(t)\mathbf{y} = \mathbf{h}_L(t), \tag{2}$$

where $\mathbf{M}_L(t)$, $\mathbf{C}_L(t)$, $\mathbf{K}_L(t)$ and $\mathbf{h}_L(t)$ are time-periodic with period T.

Equation (2) can then be expressed in the compact form as

$$\dot{\mathbf{x}} = \mathbf{P}(t)\mathbf{x} + \mathbf{f}(t) \tag{3}$$

where the state variable $\mathbf{x}$, the matrix $\mathbf{P}(t)$ and vector $\mathbf{f}(t)$ are defined by:

$$\mathbf{x} = \begin{bmatrix} \mathbf{y} \\ \dot{\mathbf{y}} \end{bmatrix}, \quad \dot{\mathbf{x}} = \begin{bmatrix} \dot{\mathbf{y}} \\ \ddot{\mathbf{y}} \end{bmatrix} \tag{4}$$

$$\mathbf{P}(t) = \begin{bmatrix} 0 & \mathbf{E} \\ -\mathbf{M}_L^{-1}\mathbf{K}_L & -\mathbf{M}_L^{-1}\mathbf{C}_L \end{bmatrix}, \ \mathbf{f}(t) = \begin{bmatrix} 0 \\ \mathbf{M}_L^{-1}\mathbf{h}_L \end{bmatrix}. \tag{5}$$

In this study, the optimal design of control parameters for linear differential systems with time-periodic coefficients is addressed. Firstly, an overview of the numerical algorithm for calculating stable conditions of linear differential systems with time-periodic coefficients is presented in Sect. 2. In the next sections, a procedure based on Taguchi method for optimal design of the stable parameters of a system described by Eq. (2) is proposed with some concluding remarks. The proposed approach is then applied to a single-link flexible manipulator that perform a simple harmonic motion.

2 Numerical Calculation of Stable Conditions of Linear Differential Systems with Time-Periodic Coefficients: A Review

Consider a system of homogeneous differential equations as

$$\dot{\mathbf{x}} = \mathbf{P}(t)\mathbf{x} \tag{6}$$

where $\mathbf{P}(t)$ is a continuous T-periodic matrix with $n \times n$. According to Floquet theory [12–16], the characteristic equation of Eq. (6) is independent from the fundamental solutions. Therefore, the characteristic equation can be formulated by the following way.

Firstly, we specify a set of n initial conditions $\mathbf{x}_i(0)$ for $i = 1, ..., n$ with the following elements

$$x_s^{(i)} = \begin{cases} 1, \ s = i \\ 0, \ s \neq i \end{cases} \tag{7}$$

and $[\mathbf{x}_1(0), \mathbf{x}_2(0), \ldots, \mathbf{x}_n(0)] = \mathbf{I}$, where $\mathbf{I}$ denote $n \times n$ identity matrix. Taking numerical integration of Eq. (6) within interval $[0, T]$ for n given initial conditions respectively, we obtain n vectors $\mathbf{x}_i(T)$, $i = 1, \ldots, n$. Matrix $\mathbf{\Phi}(t)$ defined by

$$\mathbf{\Phi}(T) = [\mathbf{x}_1(T), \mathbf{x}_2(T), \ldots, \mathbf{x}_n(T)] \tag{8}$$

is called the monodromy matrix [15] of Eq. (6). The characteristic equation of Eq. (6) can then be written in the form

$$|\mathbf{\Phi}(T) - \rho \mathbf{I}| = \begin{vmatrix} x_1^{(1)}(T) - \rho & x_2^{(1)}(T) & \dots & x_n^{(1)}(T) \\ x_1^{(2)}(T) & x_2^{(2)}(T) - \rho & \dots & x_n^{(2)}(T) \\ .. & .. & \dots & .. \\ x_1^{(n)}(T) & x_2^{(n)}(T) & \dots & x_n^{(n)}(T) - \rho \end{vmatrix} = 0 \tag{9}$$

The Eq. (9) yields a n order polynomial equation

$$\rho^n + a_1\rho^{n-1} + a_2\rho^{n-2} + \ldots + a_{n-1}\rho + a_n = 0 \tag{10}$$

Roots $\rho_i, i = 1, \ldots, n$ of Eq. (10) are called Floquet multipliers of (6). Based on these *Floquet multipliers* stability criteria of (6) are given as following:

- If the moduli of all the *Floquet multipliers* of the characteristic Eq. (10) are less than the unity, then the periodic system (6) is asymptotically stable at the origin.
- If even one of *Floquet multipliers* of the characteristic equation has a modulus larger than unity then the periodic system (6) is asymptotically unstable at the origin.
- If there is no *Floquet multiplier* of the characteristic Eq. (10) with a modulus greater than unity, but there is a *Floquet multiplier* with a modulus equal to the unity, then the solution of the system of differential Eqs. (6) may be stable, and may also be unstable, depends on the nonlinear terms.

The problem of the stability control of linear differential equation systems with time-periodic coefficients is as follows. In the general case, the solution to the characteristic Eq. (10) is a function of m parameters $u_1, u_2, \ldots, u_m$. Based on the stability criteria according to the Floquet multipliers [12, 13], we derive the definition of the parameter vector as follows.

Definition: The parameter vector

$$\mathbf{u} = [u_1\ u_2\ \ldots\ u_m]^T \tag{11}$$

of the differential equation system (6) is called asymptotic stable parameter vector if all the Floquet multipliers $\rho_k(u_1, u_2, \ldots, u_m)$ of the characteristic Eq. (10) have modulus less than unity. Conversely, if at least one Floquet multiplier of the characteristic Eq. (10).

In case the parameter vector $\mathbf{u}$ is not stable, we add a simple PD controller to the system to force it stable. The m freely selectable parameters $u_1, u_2, \ldots, u_m$ of the coefficients of the linear differential equation system (6) determine an m-dimensional solution space. It is limited in engineering so that these parameters can only be changed in a certain specified domain. Thus, there are obtained the following constraints:

$$u_{i\min} \leq u_i \leq u_{i\max}\ (i = 1, \ldots, m) \tag{12}$$

3 Dynamic Model of a Single-Link Flexible Manipulator

Considering a single-link flexible manipulator as shown in Fig. 1, link OE of length l with a payload at the free end rotates about vertical axis O. The tip mass m_E is attached at E. The link is considered as a homogeneous beam with parameters are shown in Table 1.

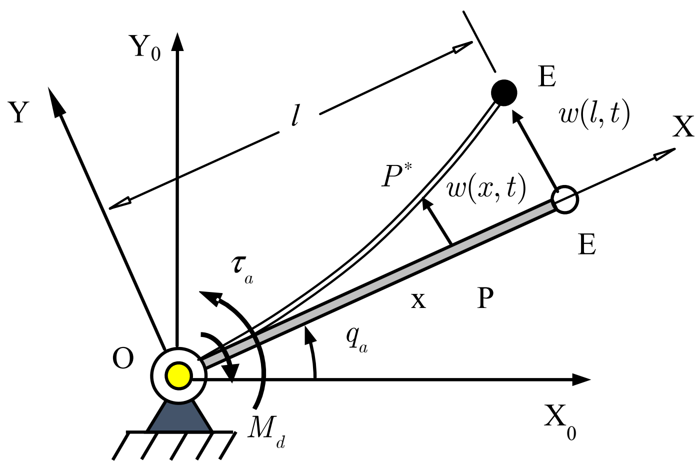

Fig. 1. Single-link flexible manipulator

Table 1. Parameters of the manipulator

Parameters of the model	Variable and unit	Value
Length of link	l (m)	0.9
Sectional area of beam	A (m^2)	4×10^{-4}
Density of beam	ρ (kg/m^3)	2700
Area moment of inertia	I (m^4) = bh^3/12	1.3333×10^{-8}
Modulus	E (N/m^2)	7.11×10^{10}
Mass moment of inertia of link 1 (including the hub)	J_1 (kg.m^2)	5.86×10^{-5}
Tip mass	m_E (kg)	0.1
Damping coefficient	α $(N.m.s/rad)$	0.01

The fundamental motion of the manipulator corresponding to applied torque $\boldsymbol{\tau}^R(t)$ is described by $\mathbf{q}^R(t)$, in which the beam is considered as a rigid link. The generalized coordinate of a manipulator is

$$\mathbf{q}^R(t) = [\, q_a^R(t)\; q_e^R(t)\,]^T = [\, q_a^R(t)\; 0\,]^T. \tag{13}$$

and the torque $\boldsymbol{\tau}^R(t)$ is

$$\boldsymbol{\tau}^R(t) = [\, \tau_a^R\; \tau_e^R\,]^T = [\, \tau_a^R\; 0\,]^T \tag{14}$$

In Eqs. (13) and (14), $q_e^R(t)$ denotes the elastic generalized coordinate and $\tau_e^R(t)$ the elastic torque of the virtual rigid link. The differential equations of a single-link flexible manipulator can be expressed in the following matrix form [16]

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) = \boldsymbol{\tau}(t) \tag{15}$$

where $\mathbf{q}$, $\dot{\mathbf{q}}$ and $\ddot{\mathbf{q}}$ are vectors of generalized coordinate, velocity and acceleration, respectively

$$\mathbf{q} = [q_a,\ q_e]^T,\ \ \boldsymbol{\tau}(t)\ =\ [\tau_a(t),\ \tau_e(t)]^T = [\tau_a(t),\ 0]^T \tag{16}$$

Let Δq_a and Δq_e be the difference between the real motion $\mathbf{q}(t)$ and the fundamental motion $\mathbf{q}^R(t)$, it follows that

$$q_a(t) = q_a^R(t) + \Delta q_a(t) = q_a^R(t) + y_1(t) \tag{17}$$

$$q_e(t) = q_e^R(t) + \Delta q_e(t) = y_2(t) \tag{18}$$

where y_1 and y_2 are called the perturbed motions. Similarly, we have

$$\boldsymbol{\tau}(t) = [\tau_a(t),\ \tau_e(t)]^T = [\tau_a(t),\ 0]^T \tag{19}$$

Substituting Eqs. (17), (18) into Eq. (15) and using Taylor series expansion around the fundamental motion, then neglecting nonlinear terms, we obtain a system of linear differential equations with time-varying coefficients for the manipulator as follows [16]

$$\mathbf{M}_L(t)\ddot{\mathbf{y}} + \mathbf{C}_L(t)\dot{\mathbf{y}} + \mathbf{K}_L(t)\mathbf{y} = \mathbf{h}_L(t). \tag{20}$$

Matrices $\mathbf{M}_L(t)$, $\mathbf{C}_L(t)$, $\mathbf{K}_L(t)$ and vector $\mathbf{h}_L(t)$ in Eq. (20) have the following form [16]

$$\mathbf{M}_L(t) = \begin{bmatrix} J_1 + m_E l^2 + \frac{1}{3} m_{OE} l^2 & \rho A D_1 + m_E l X_1(l) \\ m_E l X_1 + \rho A D_1 & m_E X_1^2(l) + \rho A m_{11} \end{bmatrix} \tag{21}$$

$$\mathbf{C}_L(t) = \begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix} \tag{22}$$

$$\mathbf{K}_L(t) = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \tag{23}$$

where

$$\begin{aligned} k_{11} &= -l \sin q_a^R(t) m_E g - \frac{m_{OE} g l \sin q_a^R(t)}{2}, \\ k_{12} &= k_{21} = -m_E g X_1(l) \sin q_a^R(t) - \mu g \sin q_a^R(t) C_1, \\ k_{22} &= -m_E[\dot{q}_a^R(t)]^2 X_1^2(l) - \rho A[\dot{q}_a^R(t)]^2 m_{11} + EI k_{11}^*. \end{aligned}$$

and

$$\mathbf{h}_L(t) = \begin{bmatrix} 0 \\ -m_E g X_1(l)\cos q_a^R(t) - \mu g \cos q_a^R(t) C_1 - m_E l X_1 \ddot{q}_a^R(t) - \rho A D_1 \ddot{q}_a^R(t) \end{bmatrix} \tag{24}$$

where fundamental motion is choosen as $q_a^R(t) = 0.5\pi + 0.5\pi\sin(\Omega t)$ and C_1, D_1, X_1, m_{11}, k_{11}^* are constants. It should be noted that matrices $\mathbf{M}_L(t)$, $\mathbf{C}_L(t)$, $\mathbf{K}_L(t)$ and vector $\mathbf{h}_L(t)$ in this example are time-periodic with period T.

In order to investigate the dynamic stability of an elastic single-link robot, we consider the homogeneous differential equation corresponding to Eq. (20)

$$\mathbf{M}_L(t)\ddot{\mathbf{y}} + \mathbf{C}_L(t)\dot{\mathbf{y}} + \mathbf{K}_L(t)\mathbf{y} = 0. \tag{25}$$

For numerical simulation, the calculating parameters of the considered manipulator are listed in Table 1.

It follows from the parameters in Table 1 that

$$C_1 = -0.704632, \quad D_1 = -0.460710,$$
$$m_{11} = 0.899850, \quad k_{11}^* = 16.955151, \quad X_1 = -1,9987$$

Some calculation results of the maximum value of the Floquet multiplier are listed in Table 2.

Table 2. Modulus of Floquet multiplier for four cases

Case 1: $\Omega = 2\pi$	$\lvert\rho_1\rvert = 13.7797,\ \lvert\rho_2\rvert = 0.0706,\ \lvert\rho_3\rvert = 0.5651,\ \lvert\rho_4\rvert = 0.5651.$
Case 2: $\Omega = 4\pi$	$\lvert\rho_1\rvert = 3.7506,\ \lvert\rho_2\rvert = 0.2628,\ \lvert\rho_3\rvert = 0.7517,\ \lvert\rho_4\rvert = 0.7517.$
Case 3: $\Omega = 6\pi$	$\lvert\rho_1\rvert = 2.4175,\ \lvert\rho_2\rvert = 0.4097,\ \lvert\rho_3\rvert = 0.8268,\ \lvert\rho_4\rvert = 0.8268.$
Case 4: $\Omega = 8\pi$	$\lvert\rho_1\rvert = 1.9396,\ \lvert\rho_2\rvert = 0.5119,\ \lvert\rho_3\rvert = 0.8674,\ \lvert\rho_4\rvert = 0.8674.$

With the initial condition

$$t = 0 : \mathbf{x}(0) = [0\ 0\ 0.25\pi\ \ 0]^T \tag{26}$$

we calculate transient vibration of the flexible manipulator with the parameters given in Table 2. Some calculation results of the transient vibration are shown in Figs. 2, 3, 4 and 5.

From Table 2 and Figs. 2, 3, 4 and 5 we can see that in the investigated cases the maximum values of the Floquet multiplier are greater than 1 and the transition oscillations tend to increase gradually. Therefore, the study of dynamic stability conditions is necessary in controlling the flexible manipulator.

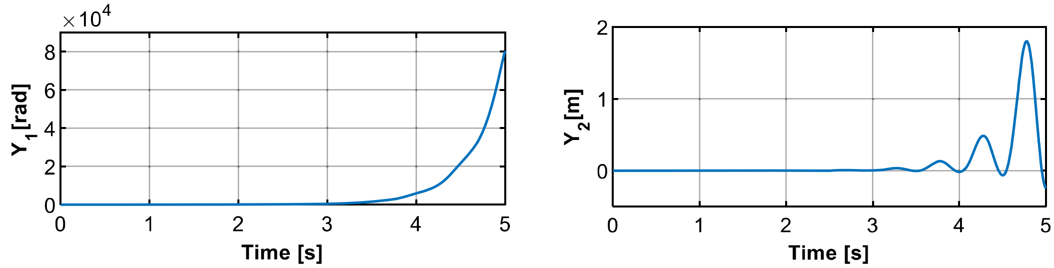

Fig. 2. Transient vibration of the flexible manipulator with $\Omega = 2\pi$

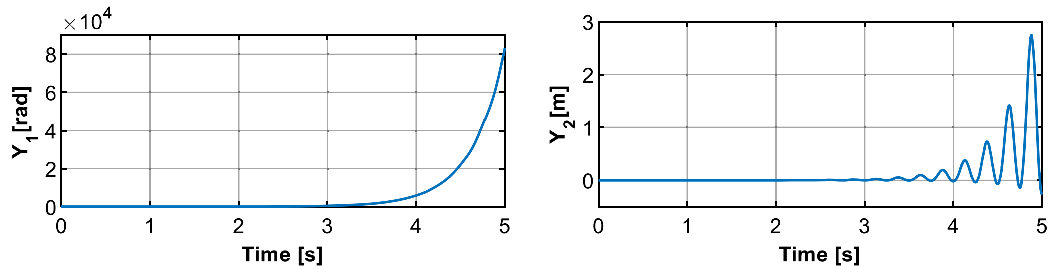

Fig. 3. Transient vibration of the flexible manipulator with $\Omega = 4\pi$

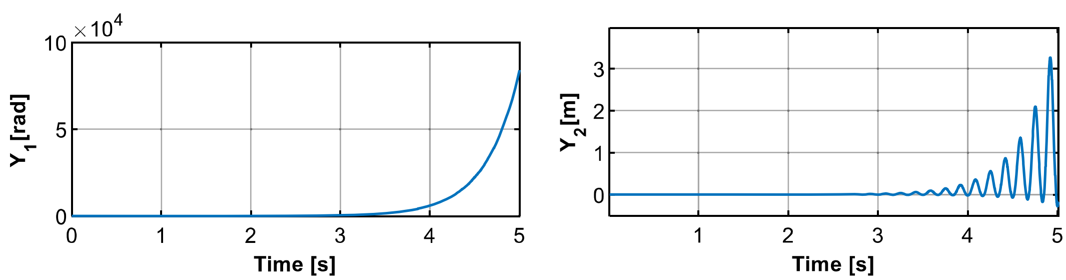

Fig. 4. Transient vibration of the flexible manipulator with $\Omega = 6\pi$

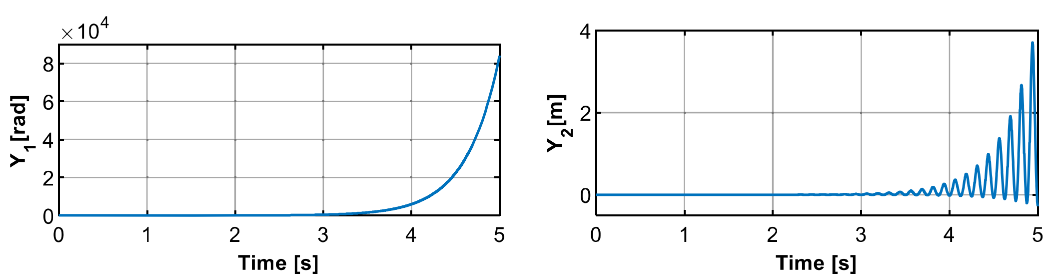

Fig. 5. Transient vibration of the flexible manipulator with $\Omega = 8\pi$

4 Dynamic Stability Control of a Flexible Manipulator Using the Taguchi Method

In the harmonic fundamental motion of a flexible manipulator, the matrices and vector of the linear differential Eq. (20) are time-periodic with period T. In this section, we propose an algorithm to control the dynamic stability of a flexible manipulator.

4.1 The PD Controller

It should be noted that for the stability control of a flexible manipulator, we can design a PD controller as follows

$$\Delta\tau_a = -k_{d1}(\dot{q}_a - \dot{q}_a^R) - k_{p1}(q_a - q_a^R) = -k_{d1}\dot{y}_1 - k_{p1}y_1 \tag{27}$$

The linearized equation according to Eq. (20) now takes the expression

$$\mathbf{M}_L(t)\ddot{\mathbf{y}} + \mathbf{C}_L(t)\dot{\mathbf{y}} + \mathbf{K}_L(t)\mathbf{y} = \mathbf{h}_L(t) - \mathbf{K}_D\dot{\mathbf{y}} - \mathbf{K}_P\mathbf{y}, \tag{28}$$

where $\mathbf{K}_D$ and $\mathbf{K}_P$ are diagonal matrices with positive elements as

$$\mathbf{K}_D = \begin{bmatrix} k_{d_1} & 0 \\ 0 & 0 \end{bmatrix};\ \mathbf{K}_P = \begin{bmatrix} k_{p_1} & 0 \\ 0 & 0 \end{bmatrix}. \tag{29}$$

It follows from Eqs. (28) that

$$\mathbf{M}_L(t)\ddot{\mathbf{y}} + [\mathbf{C}_L(t) + \mathbf{K}_D]\dot{\mathbf{y}} + [\mathbf{K}_L(t) + \mathbf{K}_P]\mathbf{y} = \mathbf{h}_L(t) \tag{30}$$

Equation (30) can then be written in the form

$$\mathbf{M}_L^{(1)}(t)\ddot{\mathbf{y}} + \mathbf{C}_L^{(1)}(t)\dot{\mathbf{y}} + \mathbf{K}_L^{(1)}(t)\mathbf{y} = \mathbf{h}_L^{(1)}(t) \tag{31}$$

where

$$\mathbf{M}_L^{(1)}(t) = \mathbf{M}_L(t),\ \mathbf{K}_L^{(1)}(t) = \mathbf{K}_L(t) + \mathbf{K}_P,\ \mathbf{C}_L^{(1)}(t) = \mathbf{C}_L(t) + \mathbf{K}_D,\ \mathbf{h}_L^{(1)}(t) = \mathbf{h}_L(t) \tag{32}$$

It should be noted that, the Eq. (31) can then be expressed in the compact form as Eq. (2). To study the dynamic stability conditions of the manipulators, the homogeneous linear differential system corresponding to Eq. (31) can be written in the following form

$$\dot{\mathbf{x}} = \mathbf{P}(t)\mathbf{x}, \tag{33}$$

where $\mathbf{P}(t)$ is a matrix of periodic elements with period T.

4.2 Determination of Gain Values According to Floquet Multipliers by the Taguchi Method

Using the Taguchi method [8, 9], Khang et al. proposed an algorithm to determine the optimal parameters of the TMDs to reduce the vibrations of the mechanical systems described by the system of linear differential equations with constant coefficients [10, 11]. The main problem in the papers [10, 11] is to calculate the eigenvalues of the constant matrix. In this paper, we use the Taguchi method to determine the gain values of the PD controller for the system the system of linear differential equations with periodic coefficients (31). The main task of the problem of determining the control parameters of the periodic system of linear differential equations is to determine the Floquet multipliers of the periodic matrix. The problem of determining the control parameters of the periodic system of linear differential equations is a new problem. The problem of determining the Floquet multipliers of a periodic matrix is much more difficult than the problem of determining the eigenvalues of a constant matrix.

The target function needs to minimized is defined by:

$$f(\mathbf{u}) = \max_i |\rho_i(\mathbf{u})| - \rho_d \rightarrow \min, \ \text{with } \mathbf{u} = [k_{p1}, k_{d1}]^T. \tag{34}$$

In which $\max_i |\rho_i(\mathbf{u})|$ is the biggest modulus of Floquet multipliers in the i^{th} experiment, and ρ_d is the target Floquet multiplier. The desired value of the target Floquet multiplier is usually chosen empirically. Some calculation results of the maximum value of the Floquet multipliers are presented in Table 3.

Table 3. Control parameters and Floquet multipliers

Ω	ρ_d	k_{p1}	k_{d1}	$\lvert\rho\rvert$
2π	0.3	37.1617	29.2410	$\lvert\rho_1\rvert = 0.3,\ \lvert\rho_2\rvert = 0,\ \lvert\rho_3\rvert = 0,\ \lvert\rho_4\rvert = 0$
4π	0.3	28.7617	11.7501	$\lvert\rho_1\rvert = 0.3,\ \lvert\rho_2\rvert = 0,\ \lvert\rho_3\rvert = 0,\ \lvert\rho_4\rvert = 0$
6π	0.3	22.2666	6.7208	$\lvert\rho_1\rvert = 0.3,\ \lvert\rho_2\rvert = 0.0042,\ \lvert\rho_3\rvert = 0,\ \lvert\rho_4\rvert = 0$
8π	0.4	23.0147	6.8628	$\lvert\rho_1\rvert = 0.4,\ \lvert\rho_2\rvert = 0.0148,\ \lvert\rho_3\rvert = 0,\ \lvert\rho_4\rvert = 0$

4.3 Simulation Results

Using the initial condition

$$t = 0 : \mathbf{x}(0) = [0\ 0\ 0.25\pi\ \ 0]^T \tag{35}$$

we can calculate transient vibration of the flexible manipulator with the parameters given in Table 3. Some calculation results of the transient vibration are shown in Figs. 6, 7, 8 and 9.

From Figs. 6, 7, 8 and 9, we can see that with the selected control parameter, the transient vibration of the flexible manipulator decreases rapidly to zero. In other words, the dynamic stability of the flexible manipulator is guaranteed by a simple PD controller.

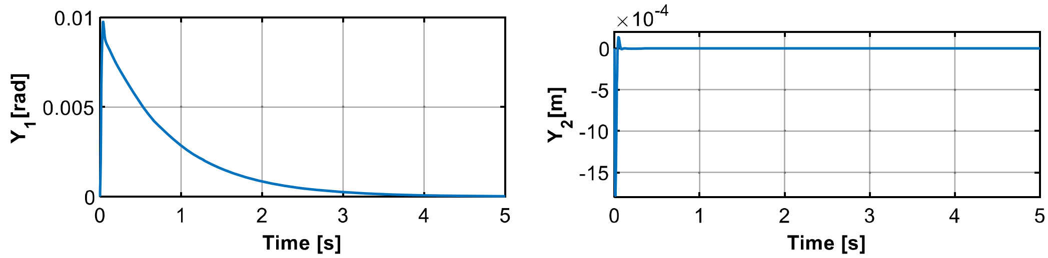

Fig. 6. Transient vibration of the flexible manipulator with control torque case $\Omega = 2\pi$

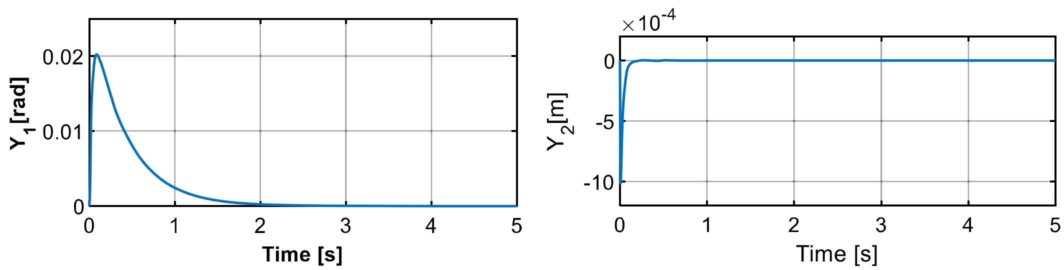

Fig. 7. Transient vibration of the flexible manipulator with control torque case $\Omega = 4\pi$

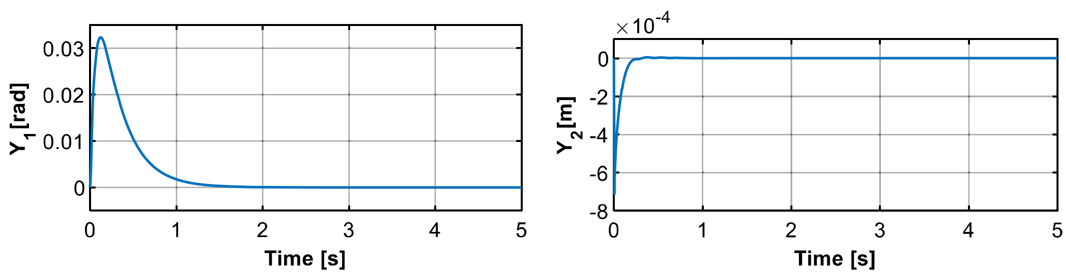

Fig. 8. Transient vibration of the flexible manipulator with control torque case $\Omega = 6\pi$

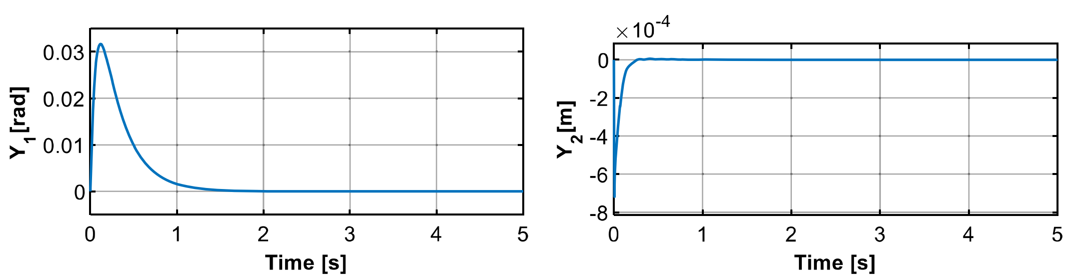

Fig. 9. Transient vibration of the flexible manipulator with control torque case $\Omega = 8\pi$

5 Conclusions

The selection of the stable parameters for the linear differential equation system is an important part of the dynamic problem for flexible manipulators. This paper presented a procedure for the optimal design of control parameters of the homogeneous linear differential equations with time-periodic coefficients. The new findings made in this study are summarized as follows:

1) Using the Taguchi method, a procedure to optimally design the stability control parameters of a system of homogeneous linear differential equations with periodic coefficients over time has been proposed.
2) Numerical calculation of the dynamic stability properties of a single-link flexible manipulator according to the Taguchi method has been implemented.
3) The method proposed in this paper can be used to calculate control parameters for multi-link flexible robots.

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