Taylor & Francis
Taylor & Francis Group

Observer-based finite-time control of linear fractional-order systems with interval time-varying delay

Nguyen T. Thanh , Piyapong Niamsup & Vu N. Phat

Taylor & Francis
Taylor & Francis Group

Check for updates

Observer-based finite-time control of linear fractional-order systems with interval time-varying delay

Nguyen T. Thanh[a], Piyapong Niamsup [b] and Vu N. Phat[c]

[a]Department of Mathematics, University of Mining and Geology, Hanoi, Vietnam; [b]RCMAM, Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai, Thailand; [c]ICRTM, Institute of Mathematics, VAST, Hanoi, Vietnam

ABSTRACT

In this paper, the problem of finite-time observer-based control for linear fractional-order systems with interval time-varying delay is studied. The delay is assumed to vary within an interval with known lower and upper bounds. A new proposition on estimating Caputo derivatives of quadratic functions is given. Based on the proposed result, delay-dependent sufficient conditions for finite-time stability and for designing state feedback controllers and observer gains for observer-based control problem are established in terms of a tractable linear matrix inequality and Mittag–Leffler function.

A numerical example with simulation is given to demonstrate the effectiveness of the proposed method.

1. Introduction

During the recent years, with the help of fractional calculus, stability and control theory of fractional-order systems has been studied by many authors due to their many useful applications in mathematical control theory (L. Chen et al., 2019; Kilbas et al., 2006; Krol, 2011). In practice, we are not only interested in asymptotic Lyapunov stability, but also in the behaviour of the system on finite-time interval called finite-time stability (FTS) (Amato et al., 2014; Dorato, 1961). FTS involves dynamical systems whose solutions do not exceed a certain bound during the specified time. While Lyapunov stability deals with the behaviour of a system at infinity time, FTS concerns solution behaviour within a finite short interval. Study of FTS for fractional-order systems (FOSs) has important value in practical applications. Many excellent results on the finite-time stability and control for FOSs have been obtained in the last five years. Based on Lyapunov direct method and generalised fractional Gronwall inequality, the finite-time stability conditions for FOSs with delays are proposed in S. Liu et al. (2016), L. Liu et al. (2017), Phat and Thanh (2018), F. Wang et al. (2017), B. Wang et al. (2016) and Wu et al. (2019). For control problems of FOSs, we can find sufficient conditions for the state feedback synchronisation of fractional-order

neural networks with delays in Zhang et al. (2018), for guaranteed cost control of nonlinear FOSs in Phat et al. (2019), for sliding mode control of singular FOSs in Meng et al. (2020) and for adaptive sliding mode control of nonlinear delay FOSs with uncertainties in Z. Wang et al. (2020).

On the other hand, problem of constructing observer-based controllers is very important when we can not access all the states of the system, or when the system output measurements do not provide a complete information on the internal system states, and then, the full-order observer-based control problem has been investigated with a growing interest in the past years (Buslowicz, 2016; Huong & Thuan, 2017; Lazarevic & Spasic, 2009; H. M. Trinh et al., 2010). It is notable that most of the authors deal with the problem of FOSs without delays or with constant delay, few authors investigate this problem of FOSs with time-varying delay. As shown in Trinh and Fernando (2012), under the practical constraints that not all of the state variables of the system are available for feedback control and the unknown time-varying delay in the full-order observer system can make it un-realisable to evaluate the observer error states, and hence the major difficulties in the design of control observers for LFOSswith delays are their appearance

of the time-varying delays and disturbances may cause the un-realizability and instability. Based on the Lyapunov function approach, numerous sufficient conditions have been obtained for the observer-based control problem of LFOSs with constant delay (Lan & Zhou, 2013; Li et al., 2014; Mathiyalagan et al., 2015). By using full-order memory state observer the authors of J. D. Chen (2007), H. Liu et al. (2012), G. Chen et al. (2013) and P. Liu (2013) proposed sufficient conditions for designing feedback controllers and observer gains for systems subjected to time-varying delay. However, the results obtained there are not only strict but also less practicable, since as shown in Trinh et al. (2012), the observer structure is un-realisable if the time-varying delay appeared in the full-order Luenbeger observer system such that we can not evaluate the unknown delayed observer. To overcome these drawbacks the authors of Yu and Lien (2007) and Thuan et al. (2012) used full-order memoryless state observers in the full-order Luenbeger observer system, i.e. the full-order observer system does not include any time-varying delay and in this paper, the full-order observer system containing known constant delay, to construct state feedback and observer controllers for LFOSs with time-varying delay. Therefore, to the best of our knowledge, the observer-based control problem for FOSs with time-varying delay has not been fully investigated in the literature and it motivates the research of our paper.

In this paper, the problem of observer-based control is studied for linear FOSs subjected to time-varying delay. Our objective is to design state feedback controllers and observer gains that stabilise the state observer error system on finite-time interval. The main contributions of our paper are the following. Firstly, we consider linear FOSs, where the time-varying delay is a bounded continuous function belonging finite-time interval. A new proposition on estimating Caputo derivative of quadratic functions is proposed to guarantee FTS of the observer error system. Secondly, to avoid the appearance of the unknown time-varying delay in the observer system, we employ a memory full-order state observer depending on the well-known delays to design the state feedback and observer controllers. Finally, using LMI technique, Laplace transformation and specific properties of Mittag-Leffler functions, delay-dependent sufficient conditions for FTS of the observer error closed-loop systems and for constructing state feedback and

observer controllers are established in terms of a linear matrix inequality and Mittag-Leffler function. A numerical example is provided to illustrate the effectiveness of the obtained results.

The paper is organised as follows. Preliminaries of fractional calculus, problem formulation and auxiliary results are reviewed in Section 2. Section 3 presents main result on observer-based control of linear time-varying delay FOSs including sufficient conditions for designing state feedback and observer controllers. In Section 4, we give a numerical example with simulation and our conclusions in Section 5.

2. Auxiliary results and preliminaries

Throughout this paper we denote by $R^{n \times r}$ the set of all matrices with dimension $(n \times r)$; by $\lambda(P)$ the set of all eigenvalues of P. For matrix P, $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ denote the maximum and minimum eigenvalues of P, respectively. $\|P\|$ denotes the spectral norm defined by $\sqrt{\lambda_{\max}(P^{\top}P)}$; by $\mathbb{P}_k$ the diagonal matrix of $k \times k$ dimensions: $\mathbb{P}_k = \text{diag}\{P, \ldots, P\}_k$; by $C[0, T]$ the set of all continuous functions on $[0, T]$; by $C([-h, 0], R^n)$ the set of all R^n-valued continuously functions on $[-h, 0]$; by $[a]$ the integer part of number a; by $L^1[a, b]$ the set of all functions integrable on $[a, b]$; by $L^2([a, b], R^m)$ the space of all R^m−valued square integrable functions on $[a, b]$; $(x, y) = x^T y$, $\|x\| = \sqrt{(x, x)}$, $|x|_P^2 = (x, Px)$; by $H^{\alpha}[0, T]$, $\alpha \in (0, 1)$, the standard Holder space consisting of continuous functions $x(t)$ on $[0, T]$ such that

$$\|x\|_{H^{\alpha}} := \max_{t \in [0,T]} |x(t)| + \sup_{0 \le s < t \le T} \frac{|x(t) - x(s)|}{(t-s)^{\alpha}} < \infty;$$

by $H_0^{\alpha}[0, T], \alpha \in (0, 1)$, the subset of $H^{\alpha}[0, T]$, consisting of functions such that

$$\max_{t \in [0,T]} |x(t)| + \sup_{0 \le s < t \le T, \, t-s \le \varepsilon} \frac{|x(t) - x(s)|}{(t-s)^{\alpha}} \to 0,$$

$$\text{as } \varepsilon \to 0.$$

Some definitions and auxiliary results on fractional calculus are introduced from Kilbas et al. (2006) as follows. The Riemann–Liouville integral $I^{\alpha}f(t)$, the Riemann–Liouville derivative $D_R^{\alpha}f(t)$, and the Caputo fractional derivative $D_C^{\alpha}f(t)$ are defined for $f \in L^1[0, T]$ as follows, respectively.

$$I^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau)}{(t-s)^{1-\alpha}} \, d\tau,$$

$$D_R^\alpha f(t) = \frac{\mathrm{d}}{\mathrm{d}t}[I^{1-\alpha}f(t)],$$

$$D_C^\alpha f(t) = D_R^\alpha[f(t) - f(0)],$$

where $\alpha \in (0,1), \Gamma(\cdot) := \int_0^\infty e^{-t}t^{s-1}\,\mathrm{d}t, s > 0$. The Mittag–Leffler function is given as

$$E_{\alpha,s}(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(n\alpha + s)}, \quad \alpha > 0,\ s > 0.$$

Letting $s = 1$ for $E_\alpha(z) := E_{\alpha,1}(z)$.

Lemma 2.1 (Kilbas et al., 2006): *The Mittag–Leffler function has the following properties:*

(i) $E_\alpha(z) \geq 1,\ z \in [0, +\infty)$.
(ii) $E_\alpha(z)$ *is increasing on* $[0, +\infty)$.

We define the Laplace transform $\mathcal{L}[f(t)](s)$ for exponentially bounded function $f(\cdot)$ as

$$\Phi(s) = \mathcal{L}[f(t)](s) = \int_0^\infty e^{-s\tau}f(\tau)\,\mathrm{d}\tau.$$

Lemma 2.2 (Kilbas et al., 2006): *We have the following properties of Laplace transform:*

(i) *The Laplace transform Caputo fractional derivative* $D_C^\alpha f(t), 0 < \alpha < 1$, *is given by*

$$\mathcal{L}[D_C^\alpha f(t)](s) = s^\alpha \mathcal{L}[f(t)](s) - s^{\alpha-1}f(0),$$

(ii)

$$\mathcal{L}[t^{\alpha-1}E_{\alpha,\alpha}(\beta t^\alpha)](s) = \frac{1}{s^\alpha - \beta}, \quad \beta > 0,\ \mathrm{Re}(z)$$

$$> |\beta|^{1/\alpha},$$

(iii) *For exponentially bounded integrable functions* $f(t), g(t)$, *and* $f * g(t) = \int_0^t f(t-s)g(s)\,\mathrm{d}s$, *we have*

$$\mathcal{L}[f * g(t)](s) = \mathcal{L}[f(t)](s) \cdot \mathcal{L}[g(t)](s),$$

Lemma 2.3 (Vainikko, 2016): *For* $0 < \alpha < 1, T > 0$, *and* $x \in C[0,T]$, *the following conditions are equivalent:*

(i) $\exists D_C^\alpha x$ *is continuous on* $C[0,T]$,
(ii)

$$\exists \lim_{t\to 0} \frac{x(t) - x(0)}{t^\alpha} = \gamma,$$

$$\sup_{0 < t \leq T} \left| \int_{\xi t}^t \frac{x(t) - x(s)}{(t-s)^{1+\alpha}}\,\mathrm{d}s \right| \to 0 \quad \text{when } \xi \to 1^-;$$

(iii) *For continuous functions* $x(t), D_C^\alpha x$, *we have* $(D_C^\alpha x)(0) = \Gamma(\alpha + 1)\gamma$, *and*

$$(D_C^\alpha x)(t) = \frac{1}{\Gamma(1-\alpha)}\left(\frac{x(t) - x(0)}{t^\alpha} \right.$$
$$\left. + \frac{\alpha}{\Gamma(1-\alpha)} \int_0^t \frac{x(t) - x(s)}{(t-s)^{1+\alpha}}\,\mathrm{d}s \right).$$

Employing similar idea from Trinh and Tuan (2018), the following result on estimating the Caputo derivative of function $x^T P x$ is proposed as follows.

Proposition 2.4: *Let* $x = x(0) + I^\alpha v, v \in C[0,T], P = P^T > 0$. *Then*, $D_C^\alpha(x(t)^T P x(t))$ *exists and continuous such that*

(i) $\displaystyle \lim_{t\to 0} \frac{|x(t)|_P^2 - |x(0)|_P^2}{t^\alpha} = 2\left(x(0), \frac{Pv(0)}{\Gamma(\alpha+1)} \right),$

(ii) $D_C^\alpha(|x(t)|_P^2)(0) = 2\Big(x(0), Pv(0) \Big),$

(iii) $\displaystyle D_C^\alpha(|x(t)|_P^2) = \frac{|x(t)|_P^2 - |x(0)|_P^2}{t^\alpha \Gamma(1-\alpha)} + \frac{\alpha}{\Gamma(1-\alpha)}$
$$\times \int_0^t \frac{|x(t)|_P^2 - |x(s)|_P^2}{(t-s)^{\alpha+1}}\,\mathrm{d}s.$$

Proof: Since $D_C^\alpha x$ is continuous we have

$$\gamma := \lim_{t\to 0} \frac{x(t) - x(0)}{t^\alpha} = \frac{v(0)}{\Gamma(\alpha+1)},$$

$$\lim_{t\to 0} \frac{|x(t)|_P^2 - |x(0)|_P^2}{t^\alpha}$$

$$= \lim_{t\to 0} \frac{\Big(x(t) - x(0), Pv(t) \Big) + \Big(x(0), P[x(t) - x(0)] \Big)}{t^\alpha}$$

$$= 2(x(0), P\gamma),$$

which shows (i). It is easy to calculate the following integral

$$\int_{\xi t}^t \frac{|x(t)|_P^2 - |x(s)|_P^2}{(t-s)^{\alpha+1}}\,\mathrm{d}s$$

$$= \int_{\xi t}^t \frac{(x(t) - x(s), 2Px(t))}{(t-s)^{\alpha+1}}\,\mathrm{d}s$$

$$-\int_{\xi t}^{t}\frac{(x(t)-x(s),P[x(t)-x(s)])}{(t-s)^{\alpha+1}}\,ds$$

$$= I_1(t,\xi) - I_2(t,\xi),$$

we derive from Lemma 2.3 that

$$|I_1(t,\xi)| = \left|\left(\int_{\xi t}^{t}(t-\tau)^{-\alpha-1}(x(t)-x(\tau))\,d\tau,\ 2Px(t)\right)\right|$$

$$\leq \left|\int_{\xi t}^{t}(t-\tau)^{-\alpha-1}(x(t)-x(\tau))\,d\tau\right|2|Px(t)|$$

$$\leq \sup_{0<t\leq T}\left|\int_{\xi t}^{t}(t-\tau)^{-\alpha-1}(x(t)-x(\tau))\,d\tau\right|2$$

$$\times \sup_{t\in[0,T]}|Px(t)|,$$

which gives $I_1(t,\xi)\to 0$ when $\xi\to 1^-$. Moreover, we have

$$x = x(0) + \gamma t^{-\alpha} + x_0,\ x_0 \in H_0^{\alpha}[0,T],\quad t\in(0,T],$$

which implies

$$\left|\frac{x(t)-x(s)}{(t-s)^{\alpha}}\right| \leq \left|\gamma\frac{t^{\alpha}-\tau^{\alpha}}{(t-s)^{\alpha}}\right| + \left|\frac{x_0(t)-x_0(s)}{(t-s)^{\alpha}}\right|$$

$$\leq \gamma\frac{(t-s)\alpha c^{\alpha-1}}{(t-s)^{\alpha}} + \left|\frac{x_0(t)-x_0(s)}{(t-s)^{\alpha}}\right|,$$

$$\leq \gamma\alpha[1/\xi - 1]^{1-\alpha}$$

$$+ \sup_{0\leq s<t\leq T,|t-s|\leq T(1-\xi)}\left|\frac{x_0(t)-x_0(s)}{(t-s)^{\alpha}}\right|$$

$$= h(\xi),$$

for $\xi t \leq s < t \leq T,\ \xi\in(0,1],c\in(s,t)$. Further, we see that function $h(\xi)$ is independent on s,t and goes to 0 when $\xi\to 1^-$, because of $|t-s|\leq|t-t\xi|\leq T(1-\xi),c\geq s\geq\xi t$, and $x_0\in H_0^{\alpha}[0,T]$. Thus, we get

$$|I_2(t,\xi)| = \int_{\xi t}^{t}\frac{(x(t)-x(s),P[x(t)-x(s)])}{(t-s)^{\alpha+1}}\,ds$$

$$\leq \int_{\xi t}^{t}\frac{(t-s)^{2\alpha}}{(t-s)^{\alpha+1}}\,ds\|P\|h(\xi)^2$$

$$= \int_{\xi t}^{t}(t-s)^{\alpha-1}\,ds\ \|P\|h(\xi)^2$$

$$= \frac{(t-t\xi)^{\alpha}}{\alpha}\|P\|h(\xi)^2 \leq \frac{T^{\alpha}(1-\xi)^{\alpha}}{\alpha}$$

$$\times \|P\|h(\xi)^2,$$

and hence

$$\sup_{0<t\leq T}\left|\int_{\xi t}^{t}(t-s)^{-\alpha-1}(|x(t)|_P^2 - |x(s)|_P^2)\,ds\right|,$$

which goes to 0, when $\xi\to 1^-$, which shows (iii). The proof is then completed by using the assertions of Lemma 2.3. ∎

For $0 < \alpha < 1$, we consider a linear fractional-order system with time-varying delay of the form

$$D_C^{\alpha}x(t) = Ax(t) + A_d x(t-\tau(t)) + Bu(t),$$

$$y(t) = Cx(t), \tag{1}$$

$$x(t) = \phi(t),\quad t\in[-h_2,0],$$

where $0<\alpha<1, x\in R^n$ is state vector, $u\in R^m$ is control vector, $y(t)\in R^k$ is measure output vector, $A;A_d\in R^{n\times n};B\in R^{n\times m}$ are given constant matrices, the initial function ϕ is continuous; the control $u(t)\in L^2([0,T],R^m)$; function $\tau(t)$ is continuous and satisfies

$$0 < h_1 \leq \tau(t) \leq h_2.$$

Associated with the observer control system (1) we consider the following full-order Luenbeger observer system

$$D_C^{\alpha}\hat{x}(t) = A\hat{x}(t) + A_d\hat{x}(t-h_2) + Bu(t)$$

$$+ L(y(t) - \hat{y}(t)),$$

$$\hat{y}(t) = C\hat{x}(t), \tag{2}$$

$$\hat{x}(\theta) = \phi(\theta),\quad \theta\in[-h_2,0],$$

and the control law will be defined by

$$u(t) = F\hat{x}(t), \tag{3}$$

where $\hat{x}(t)\in R^n$ is the observer vector, F and L are the state feedback and observer matrices with appropriate dimensions, respectively. Defining the difference between the real state $x(t)$ and the estimated state vector $\hat{x}(t): e(t) = x(t) - \hat{x}(t)$, we consider the following estimated observer system:

$$D_C^{\alpha}e(t) = [A-LC]e(t) + A_d e(t-h_2)$$

$$+ A_d x(t-\tau(t)) - A_d x(t-h_2),$$

$$D_C^{\alpha}x(t) = [A+BF]x(t) + A_d x(t-\tau(t)) - BFe(t),$$

$$e(\theta) = 0,\quad x(\theta) = \phi(\theta),\quad \theta\in[-h_2,0], \tag{4}$$

or in the following form

$$D_C^\alpha z(t) = \bar{A}z(t) + \bar{A}_d z(t - \tau(t)) + \bar{C}z(t - h_2), \quad (5)$$

where

$$z(t) = \begin{bmatrix} e(t) \\ x(t) \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} A - LC & 0 \\ -BF & A + BF \end{bmatrix},$$

$$\bar{A}_d = \begin{bmatrix} 0 & A_d \\ 0 & A_d \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} A_d & -A_d \\ 0 & 0 \end{bmatrix}.$$

The objective is to construct the feedback controller F and the observer gain L in order to finite-time stabilises system (5) in following sense.

Remark 2.1: It is known from Krol ([2011]), Kilbas and Marzan ([2014]) that under initial continuous condition $\phi(t)$, system (5) has a unique continuous solution $z(t)$ on $[0, \infty)$, which is defined as

$$z(t) = z(0) + I^\alpha \left[\bar{A}z(t) + \bar{A}_d z(t - \tau(t)) + \bar{C}z(t - h_2) \right].$$

Remark 2.2: It is worth noting that for system (1) with time-varying delay, the associated full-order observer system (2) is dependent on the bounds of the delay. In our case, the time-varying delay is interval bounded: $h_1 \le \tau(t) \le h_2$, the full-order observer system (2) can be defined by the upper and lower bound h_1, h_2. In this case, the estimated error system (5) dependent on two delays $z(t - h_1), z(t - h_2)$ can be considered as a system with multiple delays, and the stability of the system is proved similarly.

Definition 2.5 (Amato et al., [2014]): For given positive numbers c_1, c_2, T, system (5) is finite-time stable w.r.t. (c_1, c_2, T) if the following relation holds

$$\sup_{\xi \in [-h_2, 0]} \phi(\xi)^T \phi(\xi) \le c_1 \Rightarrow z(t)^T z(t) \le c_2, t \in [0, T].$$

Proposition 2.6: Let $\alpha \in (0, 1)$, $P = P^T > 0$, and let $z(t)$ be a continuous solution of system (5). Then,

$$D_C^\alpha [z(t)^T P z(t)] \le 2z(t)^T P D_C^\alpha z(t), \quad t \ge 0.$$

Proof: By Remark 2.1, solution $z(t)$ is defined by

$$z(t) = z(0) + I^\alpha a(t), \quad a(t) \in C[0, \infty),$$

where $\quad a(t) = \bar{A}z(t) + \bar{A}_d z(t - \tau(t)) + \bar{C}z(t - h_2)$. Using Proposition 2.4, we have

$$D_C^\alpha (|z(t)|_P^2) - 2(z(t), P D_C^\alpha z(t)) = 0, \text{ at } t = 0, \quad \text{and}$$

$$D_C^\alpha (|z(t)|_P^2) - 2(z(t), P D_C^\alpha z(t)) = -\frac{|z(t) - z(0)|_P^2}{t^\alpha \Gamma(1 - \alpha)}$$

$$- \frac{\alpha}{\Gamma(1 - \alpha)} \int_0^t \frac{|z(t) - z(\tau)|_P^2}{(t - \tau)^{\alpha + 1}} \, \mathrm{d}\tau \le 0.$$

■

3. Design of feedback and observer controllers

In this section, we will give sufficient conditions for designing feedback control matrix F and observer gain L for system (5).

Theorem 3.1: *For given positive numbers c_1, c_2, T, system (5) is finite-time stable w.r.t. (c_1, c_2, T) if there exist a symmetric matrix $P > 0$, a matrix Q such that*

$$\begin{pmatrix} \Omega_{11} & \Omega_{12} & . & . & . & \Omega_{16} \\ * & \Omega_{22} & . & . & . & \Omega_{26} \\ . & . & . & . & . & . \\ * & * & . & . & . & \Omega_{66} \end{pmatrix} < 0, \quad (6)$$

$$\frac{\lambda_{\max}(P^{-1})}{\lambda_{\min}(P^{-1})} E_\alpha(h_2 T^\alpha) \sum_{j=0}^{[T/h_1]+1} (E_\alpha(h_2 T^\alpha) - 1)^j \le \frac{c_2}{c_1}.$$

$$(7)$$

Moreover, the feedback control matrix is $F = QP^{-1}$, the observer gain matrix is $L = 0.5PC^T$, where

$$\Omega_{11} = AP + PA^T + BQ + (BQ)^T - h_2 P,$$

$$\Omega_{14} = -QP,$$

$$\Omega_{12} = A_d P, \quad \Omega_{13} = \Omega_{15} = \Omega_{16} = 0,$$

$$\Omega_{22} = \Omega_{33} = \Omega_{55} = \Omega_{66} = -0.5 h_2 P,$$

$$\Omega_{44} = AP + PA^T + \lambda_{\min}(C^T C)(I - 2P) - h_2 P,$$

$$\Omega_{24} = PA_d^T, \quad \Omega_{34} = -PA_d^T, \quad \Omega_{46} = A_d P.$$

Proof: Setting $\mathcal{P} = P^{-1}$, we consider the quadratic function

$$V(t) := V(t, z(t)) = x(t)^T \mathcal{P} x(t) + e(t)^T \mathcal{P} e(t).$$

Taking the derivative of $V(\cdot)$ and using Proposition 2.6 gives

$$D_C^\alpha V(t) \le 2x(t)^T \mathcal{P} D^\alpha x(t) + 2e(t)^T \mathcal{P} D^\alpha e(t)$$

$$= 2x(t)^T \mathcal{P}([A + BF]x(t) + A_d x_\tau(t) - BFe(t))$$

$$+ 2e(t)^T \mathcal{P}([A - LC]e(t) + A_d e(t - h_2)$$

$$+ A_d x_\tau(t) - A_d x(t - h_2))$$

$$= 2x(t)^T \mathcal{P}([A + BQ\mathcal{P}]x(t) + A_d x_\tau(t))$$
$$- BQ\mathcal{P}e(t))$$
$$+ 2e(t)^T \mathcal{P}([A - 1/2\mathcal{P}C^T C]e(t)$$
$$+ A_d e(t - h_2) + A_d x_\tau(t)) - A_d x(t - h_2))$$
$$- \frac{1}{2}h_2 x_\tau(t)^T \mathcal{P}x_\tau(t) + \frac{1}{2}h_2 x_\tau(t)^T \mathcal{P}x_\tau(t)$$
$$- \frac{1}{2}h_2 e_\tau(t)^T \mathcal{P}e_\tau(t)$$
$$- h_2 x(t)^T \mathcal{P}x(t) - h_2 e(t)^T \mathcal{P}e(t)$$
$$+ \frac{1}{2}h_2 e_\tau(t)^T \mathcal{P}e_\tau(t)$$
$$+ h_2 V(t) + \frac{1}{2}h_2 V(t - h_2)$$
$$- \frac{1}{2}h_2 x^T(t - h_2)\mathcal{P}x(t - h_2)$$
$$- \frac{1}{2}h_2 e(t - h_2)^T \mathcal{P}e(t - h_2)$$
$$= \eta(t)^T \bar{\Omega}\eta(t) + h_2 V(t)$$
$$+ \frac{1}{2}h_2 V(t - \tau(t)) + \frac{1}{2}h_2 V(t - h_2), \quad (8)$$

where $\eta(t) = [x(t)^T \; x_\tau(t)^T \; x(t - h_2)^T \; e(t)^T \; e_\tau(t)^T \; e(t - h_2)^T]^T$, $x_\tau(t) := x(t - \tau(t))$, $e_\tau(t) := e(t - \tau(t))$ and

$$\bar{\Omega} = (\bar{\Omega}_{ij})_{6 \times 6},$$
$$\bar{\Omega}_{11} = \mathcal{P}A + A^T \mathcal{P} + \mathcal{P}[BQ + (BQ)^T]\mathcal{P} - h_2 \mathcal{P},$$
$$\bar{\Omega}_{12} = \mathcal{P}A_d, \quad \bar{\Omega}_{44} = \mathcal{P}A + A^T \mathcal{P} - C^T C - h_2 \mathcal{P},$$
$$\bar{\Omega}_{22} = \bar{\Omega}_{33} = \bar{\Omega}_{55} = \bar{\Omega}_{66} = -0.5h_2 \mathcal{P},$$
$$\bar{\Omega}_{13} = \bar{\Omega}_{15} = \bar{\Omega}_{16} = 0,$$
$$\bar{\Omega}_{23} = \bar{\Omega}_{25} = \bar{\Omega}_{26} = \bar{\Omega}_{35} = 0,$$
$$\bar{\Omega}_{36} = \bar{\Omega}_{45} = \bar{\Omega}_{56} = 0, \quad \bar{\Omega}_{14} = -\mathcal{P}[BQ]\mathcal{P},$$
$$\bar{\Omega}_{24} = A_d^T \mathcal{P}, \quad \bar{\Omega}_{34} = -A_d^T \mathcal{P}, \quad \bar{\Omega}_{46} = \mathcal{P}A_d.$$

Using condition (6), we get $\mathbb{P}_6 \bar{\Omega}\mathbb{P}_6 \leq 0$, which leads to $\bar{\Omega} \leq 0$. Moreover, from (8) it follows that

$$D_C^\alpha V(t) \leq h_2 V(t) + \frac{1}{2}h_2 V(t - \tau(t)) + \frac{1}{2}h_2 V(t - h_2).$$

Letting $N(t) = D_C^\alpha(V(t)) - h_2 V(t)$, we see that

$$N(t) \leq \frac{1}{2}h_2 V(t - \tau(t)) + \frac{1}{2}h_2 V(t - h_2)),$$

and hence

$$\sup_{0 \leq s \leq t} N(s) \leq h_2 \sup_{-h_2 \leq s \leq t - h_1} V(s),$$

because of $-h_2 \leq s - \tau(s) \leq t - h_1, \; \forall \; s \in [0, t]$. Applying the Laplace transform for the equation

$$D_C^\alpha V(t) = h_2 V(t) + N(t),$$

we obtain that

$$s^\alpha \mathcal{V}(s) - V(0)s^{\alpha-1} = h_2 \mathcal{V}(s) + \mathcal{N}(s),$$

where $\mathcal{V}(s) = \mathcal{L}[V(t)](s), \mathcal{N}(s) = \mathcal{L}[N(t)](s)$, and we get

$$\mathcal{V}(s) = (s^\alpha - h_2)^{-1}\left(V(0)s^{\alpha-1} + \mathcal{N}(s)\right). \quad (9)$$

In the sequel, the following inequalities will be used in the estimation of $V(t)$:

$$(t - s)^{\alpha-1}E_{\alpha,\alpha}(h_2(t - s)^\alpha) \geq 0, \quad t \geq 0, \; s \in [0, t],$$
$$h_2 > 0,$$
$$\int_0^t \frac{E_{\alpha,\alpha}(h_2(t - s)^\alpha)}{(t - s)^{1-\alpha}} \, ds = \frac{1}{h_2}[E_\alpha(h_2 t^\alpha) - 1].$$

Applying the inverse Laplace transform to the inequality (9), we obtain for all $t \geq 0$ that

$$V(t) = V(0)E_\alpha(h_2 t^\alpha) + \int_0^t N(s)(t - s)^{\alpha-1}$$
$$\times E_{\alpha,\alpha}(h_2(t - s)^\alpha) \, ds$$
$$\leq V(0)E_\alpha(h_2 t^\alpha) + \sup_{0 \leq s \leq t} N(s) \int_0^t (t - s)^{\alpha-1}$$
$$\times E_{\alpha,\alpha}(h_2(t - s)^\alpha) \, ds$$
$$\leq V(0)E_\alpha(h_2 t^\alpha) + \left(E_\alpha(h_2 t^\alpha) - 1\right)$$
$$\times \sup_{-h_2 \leq s \leq t - h_1} V(s).$$

By Lemma 2.1, the function $E_\alpha(h_2 t^\alpha)$ on $[0, T]$ is increasing. Hence, we have

$$V(s) \leq \sup_{\xi \in [-h_2, 0]} \phi(\xi)^T \mathcal{P}\phi(\xi)E_\alpha(h_2 t^\alpha),$$

for $-h_2 \leq s \leq 0$, and

$$V(s) \leq \sup_{\xi \in [-h_2, 0]} \phi(\xi)^T \mathcal{P}\phi(\xi)E_\alpha(h_2 t^\alpha)$$
$$+ \left(E_\alpha(h_2 t^\alpha) - 1\right) \sup_{-h_2 \leq \xi \leq t - h_1} V(\xi),$$

for $0 \le s \le t$, which leads to

$$\sup_{-h_2 \le s \le t} V(s) \le \sup_{\xi \in [-h_2, 0]} \phi(\xi)^T \mathcal{P} \phi(\xi) E_\alpha(h_2 t^\alpha)$$
$$+ \left(E_\alpha(h_2 t^\alpha) - 1 \right) \sup_{-h_2 \le \xi \le t - h_1} V(\xi).$$

Setting $R(t) = \sup_{-h_2 \le s \le t} V(s)$, for $t \in [0, T]$ we have

$$R(t) \le E_\alpha(h_2 T^\alpha) R(0) + \left(E_\alpha(h_2 T^\alpha) - 1 \right) R(t - h_1).$$

Note that $R(t)$ is an increasing function and $E_\alpha(h_2 T^\alpha) > 1$, by induction we can easily derive the following condition

$$R(t) \le R(0) E_\alpha(h_2 T^\alpha) \sum_{j=0}^{[T/h_1]+1} \left(E_\alpha(h_2 T^\alpha) - 1 \right)^j,$$

and hence

$$\lambda_{\min}(\mathcal{P}) \| z(t) \|^2 \le R(t) \le R(0) q$$
$$\le \sup_{s \in [-h_2, 0]} \phi(s)^T P^{-1} \phi(s) q,$$

where $q = E_\alpha(h_2 T^\alpha) \sum_{i=0}^{[T/h_1]+1} (E_\alpha(h_2 T^\alpha) - 1)^i$. Moreover, since

$$\phi(s)^T \mathcal{P} \phi(s) \le \lambda_{\max}(\mathcal{P}) \| \phi(s) \|^2 \le \lambda_{\max}(\mathcal{P}) c_1,$$

from condition (7) it follows that

$$\| z(t) \|^2 \le \frac{\lambda_{\max}(\mathcal{P})}{\lambda_{\min}(\mathcal{P})} q c_1 \le c_2, \quad t \in [0, T].$$

The proof is completed. ∎

Remark 3.1: It is well known that the stability of linear FOSs can be proved using the second Lyapunov function approach proposed in Camacho et al. (2014), where the authors introduced a new property on Caputo fractional derivatives of differentiable function $\| x \|^2$ (Lemma 1, p. 2953). However, for FOSs with delay, this method can not be applicable since its solutions of the system with delay are, in general, not differentiable. Therefore, Proposition 2.6 related to the estimation of the Caputo fractional derivative of quadratic functions is effectively introduced for studying the stability of linear FOSs with delay, where its solution $x(t)$ is continuous.

Remark 3.2: Note that in the proof of Theorem 3.1, we use a simple delay-independent Lyapunov function $V(x(t))$, which allows us to avoid the use of the differentiability of the delay function $\tau(t)$. Then, we do not need to employ the fractional Lyapunov stability theorem for FOSs with delay, where the Lyapunov function is necessary to be positive definite (S. Liu et al., 2017; Wen et al., 2008).

4. A numerical example

In this section, an illustrative example is given to demonstrate the validity and effectiveness of the proposed method.

Example 4.1: Consider system (1), where $\alpha = \frac{1}{2}$, the time-varying delay $\tau(t) = 0.1 |\sin(t)| + 0.8$, and

$$A = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0.1 \\ 0 & 1 \end{bmatrix},$$
$$A_d = \begin{bmatrix} 1 & 0 \\ 0.1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 4 & 0 \\ 0.1 & 3 \end{bmatrix},$$

Note that the delay function $\tau(t)$ is continuous and not differentiable. Therefore, the observer control design methods proposed in J. D. Chen (2007), H. Liu et al. (2012), G. Chen et al. (2013), and P. Liu (2013) are not applicable to this system. For $h_1 = 0.8, h_2 = 0.9$, the LMI (6) is feasible with the following solutions

$$P = \begin{bmatrix} 2.4559 & -0.0702 \\ -0.0702 & 2.9061 \end{bmatrix},$$
$$Q = \begin{bmatrix} -1.6351 & 0.8923 \\ -0.0723 & -10.8698 \end{bmatrix}.$$

The feedback control matrix F and the observer gain matrix L are defined as

$$F = QP^{-1} = \begin{bmatrix} -0.6575 & 0.2912 \\ -0.1365 & -3.7437 \end{bmatrix}$$
$$L = 0.5PC^T = \begin{bmatrix} 4.9117 & 0.0174 \\ -0.1405 & 4.3556 \end{bmatrix}.$$

Besides, for $c_1 = 0.01, c_2 = 2.7, T = 1.5$, we have

$$\frac{\lambda_{\max}(\mathcal{P})}{\lambda_{\min}(\mathcal{P})} E_\alpha(h_2 T^\alpha) \sum_{j=0}^{[T/h_1]+1} (E_\alpha(h_2 T^\alpha) - 1)^j$$
$$\times c_1 = 2.6472 < c_2 = 2.7.$$

Therefore, by Theorem 3.1, the error system (5) is finite-time stable w.r.t. (0.01, 2.7, 1.5).

Figures 1–3 show the state response of $\| z(t) \|^2$, $\| x(t) \|^2$ and $\| e(t) \|^2$ with the initial condition $\phi(\theta) = [0.09, 0]^T$, $\theta \in [-0.9, 0]$, respectively.
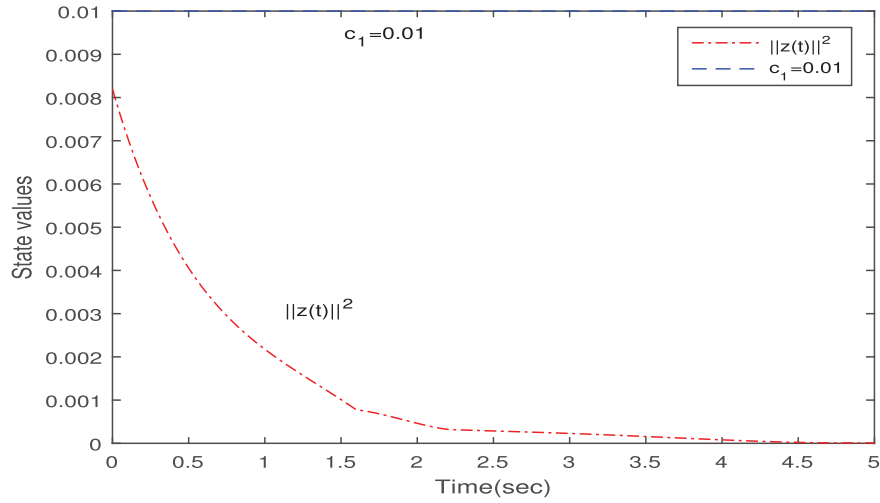
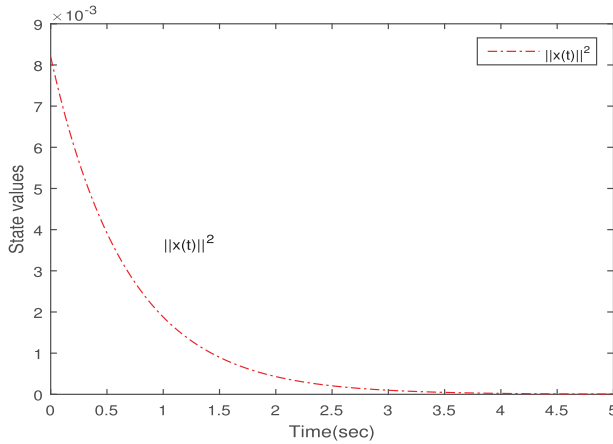**Figure 1.** The state response of $\|z(t)\|^2$.

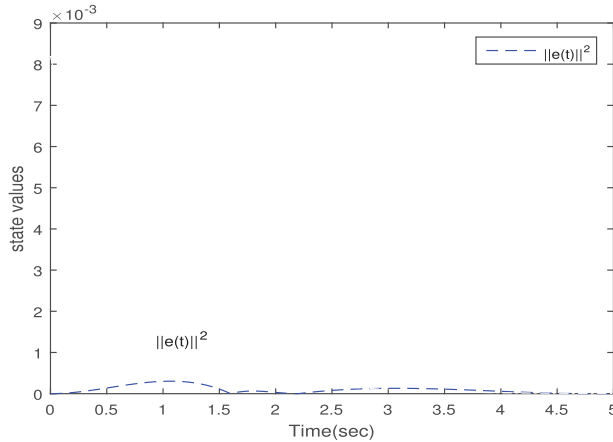

Figure 2. The state response of $\|x(t)\|^2$.

Figure 3. The state response of $\|e(t)\|^2$

5. Conclusion

We have investigated the problem of finite-time observer-based control for linear FOSs subjected to time-varying delay. To derive the main result, we have proved a new result on Caputo fractional derivative of some quadratic function. Improved delay-dependent conditions for finite-time stability of the observer closed-loop systems are established in terms of a linear matrix inequality and the Mittag–Leffler function. The obtained stability conditions are applied to construct full-order observer and feedback controllers. An example is provided to illustrate the theoretical result.

Disclosure statement

No potential conflict of interest was reported by the author(s).

Notes on contributors

Nguyen T. Thanh received the B.Sc. and Ph.D. degrees in mathematics from Hanoi National University in 2004 and 2015, respectively. He is now a senior lecturer and associate professor at Department of Mathematics, Hanoi Mining and Geology University. His research interests include stability analysis, fractional calculus, control theory and applications.

Piyapong Niamsup received the B.Sc. degree in mathematics from Chiang Mai University, Thailand in 1992. He also received the M.Sc. and Ph.D. degrees in mathematics from University of Illinois at Urbana-Champaign, USA in 1995 and 1997, respectively. In 1997, he joined the Department of Mathematics, Faculty of Science, Chiang Mai University, Thailand as a lecturer, where he became an associate professor in 2006. His research interests include complex dynamics, stability theory, switched systems, chaos synchronization, and discrete-time events.

Vu N. Phat received the B.Sc. and Ph.D. degrees in mathematics at the former USSR Bacu State University, USSR in 1975 and 1984, respectively. He received the D. Sc. Degree in mathematics at the Institute of Mathematics, Polish Academy of Sciences, Poland in 1995. Currently, he works as a professor at the Institute of Mathematics, Vietnam Academy of Science and Technology, Vietnam. He is the author/co-author of two monographs and more than 100 refereed journal papers. His research interests include systems and control theory, optimization techniques, stability analysis, fractional calculus, and time-delay systems.

ORCID

Piyapong Niamsup http://orcid.org/0000-0003-2616-8605

References

Amato, F., Ambrosino, R., Ariola, M., & Cosentino, C (2014). *Finite-time stability and control*. Springer.

Buslowicz, M. (2016). Stability conditions for linear continuous-time fractional-order state-delayed systems. *Bulletin of the Polish Academy of Sciences*, 64(1), 1–7. https://doi.org/10.1515/bpasts-2016-0001

Camacho, N. A., Duarte-Mermouda, M. A., & Gallegos, J. A. (2014). Lyapunov functions for fractional order systems. *Communications in Nonlinear Science and Numerical Simulation*, 19(9), 2951–2957. https://doi.org/10.1016/j.cnsns.2014.01.022

Chen, J. D. (2007). Robust H_∞ output dynamic observer-based control design of uncertain neutral systems. *Journal of Optimization Theory and Applications*, 132(1), 193–211. https://doi.org/10.1007/s10957-006-9143-3

Chen, G., Xiang, Z., & Karimi, H. R. (2013). Observer-based robust H_∞ control for switched stochastic systems with time-varying delay. *Abstract and Applied Analysis*, 2013, Article ID: 320703, 1–12. https://doi.org/10.1155/2013/320703

Chen, L., Li, T., Chen, Y. Q, Wu, R., & Ge, S. (2019). Robust passivity and feedback passification of a class of uncertain fractional-order linear systems. *International Journal of Systems Science*, 50, 1149–1162. https://doi.org/10.1080/00207721.2019

Dorato, P. (1961). Short-time stability in linear time-varying systems. *IRE International Convention Record*, 4, 83–87. https://doi.org/10.1109/ssst.1994.287864

Huong, D. C., & Thuan, M. V. (2017). Design of unknown-input reduced-order observers for a class of nonlinear fractional-order time-delay systems. *International Journal of Adaptive Control and Signal Processing*, 32, 412–423. https://doi.org/10.1002/acs.v32.3

Kilbas, A. A., & Marzan, S. A. (2014). Cauchy problem for differential equation with caputo derivative. *An International Journal for Theory and Applications*, 7, 297–321. http://eudml.org/doc/11247

Kilbas, A. A., Srivastava, H., & Trujillo, J. (2006). *Theory and applications of fractional differential equations*. Elsevier.

Krol, K. (2011). Asymptotic properties of fractional delay differential equations. *Applied Mathematics and Computation*, 218, 1515–1532. https://doi.org/10.1016/j.amc.2011.04.059

Lan, Y. H., & Zhou, Y. (2013). Non-fragile observer-based robust control for a class of fractional-order nonlinear systems. *Systems & Control Letters*, 62, 1143–1150. https://doi.org/10.1016/j.sysconle.2013.09.007

Lazarevic, M. P., & Spasic, A. M. (2009). Finite-time stability analysis of fractional-order time-delay systems: Gronwall's approach. *Mathematical and Computer Modelling*, 49, 475–481. https://doi.org/10.1016/j.mcm.2008.09.011

Li, C., Wang, J., Lu, J., & Ge, Y. (2014). Observer-based stabilisation of a class of fractional order non-linear systems for $0 < \alpha < 2$ case. *IET Control Theory & Applications*, 8, 1238–1246. https://doi.org/10.1049/iet-cta.2013.1082

Liu, H., Shena, Y., & Zhao, X. (2012). Delay-dependent observer-based H_∞ finite-time control for switched systems with time-varying delay. *Nonlinear Analysis: Hybrid Systems*, 6, 885–898. https://doi.org/10.1016/j.nahs.2012.03.001

Liu, L., Cao, X., Fu, Z., & Song, S. (2017). Guaranteed cost finite-time control of fractional-order positive switched systems. *Advances in Mathematical Physics*, 2017, 1–11. https://doi.org/10.1155/20172695894

Liu, P. (2013). Observer-based control for time-varying delay systems with delay-dependence. *Universal Journal of Electrical and Electronic Engineering*, 1, 31–40. https://doi.org/10.13189/ujeee.2013.010204

Liu, S., Jiang, W., Li, X., & Zhou, X. F. (2016). Lyapunov stability analysis of fractional nonlinear systems. *Applied Mathematics Letters*, 51, 13–19. https://doi.org/10.1016/j.aml.2015.06.018

Liu, S., Zhou, X. F., Li, X., & Jiang, W. (2017). Asymptotical stability of Riemann-Liouville fractional singular systems with multiple time-varying delays. *Applied Mathematics Letters*, 65, 32–39. https://doi.org/10.1016/j.aml.2016.10.002

Mathiyalagan, K., Park, J. H., & Sakthivel, R. (2015). Observer-based dissipative control for networked control systems: a switched system approach. *Complexity*, 21, 297–308. https://doi.org/10.1002/cplx.v21.2

Meng, B., Wang, X., Zhang, Z., & Wang, Z. (2020). Necessary and sufficient conditions for normalization and sliding mode control of singular fractional-order systems with uncertainties. *Science China Information Sciences*, 63, Article ID: 152202, 1–10. https://doi.org/10.1007/s11432-019-1521-5

Phat, V. N., & Thanh, N. T. (2018). New criteria for finite-time stability of nonlinear fractional-order delay systems: A

Gronwall inequality approach. *Applied Mathematics Letters*, *83*, 169–175. https://doi.org/10.1016/j.aml.2018.03.023

Phat, V. N., Thuan, M. V., & Tuan, T. N. (2019). New criteria for guaranteed cost control of nonlinear fractional-order delay systems: A Razumikhin approach. *Vietnam Journal of Mathematics*, *47*, 403–415. https://doi.org/10.1007/s10013-018-0323-x

Thuan, M. V., Phat, V. N., & Trinh, H. (2012). Observer-based controller design of time-delay systems with interval time-varying delay. *International Journal of Applied Mathematics and Computer Science*, *22*, 921–927. https://doi.org/10.2478/v10006-012-0068-8

Trinh, H., & Fernando, T (2012). *Functional observer for dynamical systems*. Springer.

Trinh, H., & Tuan, H. T. (2018). Stability of fractional-order nonlinear systems by Lyapunov direct method. *IET Control Theory & Applications*, *12*, 2417–2422. https://doi.org/10.1049/iet-cta.2018.5233

Trinh, H. M., Teh, P. S., & Fernando, T. L. (2010). Time-delay systems: Design of delay-free and low-order observers. *IEEE Transactions on Automatic Control*, *55*, 2434–2438. https://doi.org/10.1109/TAC.2010.2063510

Vainikko, G. (2016). Which functions are fractionally differentiable. *Zeitschrift für Analysis und Ihre Anwendungen*, *35*, 465–487. https://doi.org/10.4171/ZAA

Wang, B., Ding, J. L., Wu, F. J., & Zhu, D. (2016). Robust finite-time control of fractional-order nonlinear systems via frequency distributed model. *Nonlinear Dynamics*, *85*, 2133–2142. https://doi.org/10.1007/s11071-016-2819-9

Wang, F., Chen, D., Zhang, X., & Wu, Y. (2017). Finite-time stability of a class of nonlinear fractional-order system with the discrete time delay. *International Journal of Systems Science*, *48*(5), 984–993. https://doi.org/10.1080/00207721.2016.1226985

Wang, Z., Wang, X., Xia, J., Shen, H., & Meng, B. (2020). Adaptive sliding mode output tracking control based-FODOB for aclass of uncertain fractional-order nonlinear time-delayed systems. *Science China Technological Sciences*. https://doi.org/10.1007/s11431-019-1476-4

Wen, X. J., Wu, Z. M., & Lu, J. G. (2008). Stability analysis of a class of nonlinear fractional order systems. *IEEE Transactions on Circuits and Systems II: Express Briefs*, *55*, 1178–1182. https://doi.org/10.1109/TCSII.2008.2002571

Wu, F., Li, F., Chen, P., & Wang, B. (2019). Finite-time control for a fractional-order non-linear HTGS. *IET Renewable Power Generation*, *13*, 633–639. https://doi.org/10.1049/iet-rpg.2018.5734

Yu, K. W., & Lien, C. H. (2007). Delay-dependent conditions for guaranteed cost observer-based control of uncertain neutral systems with time-varying delays. *IMA Journal of Mathematical Control and Information*, *24*, 383–394. https://doi.org/10.1093/imamci/dnl031

Zhang, J., Wu, J., Bao, H., & Cao, J. (2018). Synchronization analysis of fractional-order three-neuron BAM neural networks with multiple time delays. *Applied Mathematics and Computation*, *339*, 441–450. https://doi.org/10.1016/j.amc.2018.06.013