PART I

THE INTERIOR EXTERIOR APPROACH FOR LINEAR PROGRAMMING PROBLEM

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Abstract.
In this paper we present a new interior exterior algorithm for solving linear programming problem which can be viewed as a variation of simplex method in combination with interior approach. With the assumption that a feasible interior solution to the input system is known, this algorithm uses it and appropriate constraints of the system to construct a sequence of the so called station cones whose vertices tend very fast to the solution to be found. The computational experiments show that the number of iterations of the interior exterior algorithm is significantly smaller than that of the second phase of the simplex method. Additionally, when the number of variables and constraints of the problem increase, the number of iterations of the interior exterior approach increase in a slower manner than that of the simplex method.

Keywords: Linear programming, simplex method, station cone.

1. Introduction

After its discovery by Dantzig in 1947 [6] the simplex method was unrivaled, until the late 1980s, for its utility in solving practical linear programming problems. The computational experiments show a remarkable fact that the simplex method typically requires at most 2m to 3m pivots to obtain optimality [2,3,6,7]. Although the simplex method is quite efficient in practice, there exists a class of linear programming problems for which the simplex method takes an exponential number of steps [10].
In 1979 [9] Khachian introduced the ellipsoid method which gives a bound of $O(n^5L)$ arithmetic operations on number with $O(nL)$ digits. Khachian's algorithm was of landmark importance for establishing the polynomial time solvability of linear programs. Despite its major theoretical advance, the ellipsoid method had little practical impact as the simplex method is more efficient for many classes of linear programming problems [1,14].
In 1984 [8] Kamarkar proposed a new projective method for linear programming problems which not only improved Khachian's theoretical worst-case polynomial bound but in fact promised dramatically practical performance improvement over simplex method. Karmarkar's algorithm requires $O(n^{3.5}L)$ operations on th $O(L)$ digit numbers as compared f $O(n^6L)$ such operations for the Khachian's ellipsoid method. Karmarkar's algorithm falls within the class of interior point methods. In contrast to the simplex method, which finds the optimal solution among the vertices of the feasible set, the interior point method moves through the interior of the feasible region and reaches the optimal solution only asymptotically. Stimulated by Karmarker's algorithm a varietyof interior point methods were developed for linear programming [12,16].

There are several important open problems in the theory of linear programming, the solution of which would represent fundamental breakthrough in mathematics. In the recent survey on linear programming [15] M.J. Todd has mentioned some unsolved problems: Is there a polynomial pivot rule for the simplex method? Does the bounded Hirsch conjecture hold? The immense efficiency of the simplex method in practice, despite its exponential time theoretical performance, hints that there may be variations of simplex algorithm that run in polynomial time.

In this paper we present a new interior exterior algorithm for solving linear programming problems which can be viewed as a variation of the simplex method in combination with the interior approach.We assume that the linear programming problem has a initial strict interior point *O*. Then using this point *O* we construct a sequence of the so called station cones whose vertices will allow a very fast optimal solution to be found. The new interior exterior algorithm has been tested, using MatLab, on a set of randomly generated linear problems. The computational experiments show that the number of iterations of the interior exterior approach is significantly smaller than that of the second phase of the simplex method.
The paper is organized as follows. In section 2 we introduce some results which are necessary for the construction of the algorithm. In section 3, we describe the criterion of selecting the leaving variables. The section 4 presents the main idea of the algorithm and proposes the selecting rule for entering vectors. The algorithm is presented in section 5. A numerical example have been illustatived in section 6. The section 7 presents the computational experiments for some class of small and medium size problems. Finally, some conlusions have been made in section 8.

2. Station Cone

We consider a linear programming problem in the matrix form

$$\begin{aligned} &\max\ \langle c,x\rangle \\ &x\in P:=\{x \mid Ax\le b,\ x\ge 0\}, \end{aligned} \tag{2.1}$$

where $c \in R^n, A \in R^{mxn}, b \in R^m, \forall x \in R^n$. Let $A_1, A_2, \dots, A_m$ denote the row vectors.Through this paper we suppose that (2.1) and its dual problem are nondegenerated .We also suggest the feasible region *P* of (2.1) has strict interior points. For simplicity of argument, we assume that the matrix A has full column rank *n* and *n* < *m.*

Let $I_n=\{i_1,i_2,\dots,i_n\}\subset\{1,2,\dots,m\}$ such that the vectors $A_i,\ i\in I_n$ are linear independent. This means the vector $A_i,\ i\in I_n$ establish a basis of R^n. Therefore any vector $A_l\in\mathbb{R}^n$ can be expressed as a linear combination of the vectors $A_i, i \in I_n$. Let λ_{li_k} be the linear coefficient of the vector A_l in the basis $A_{i_k},\ i_k\in I_n$, then

$$a_{lj}=\sum_{k=1}^{n}\lambda_{li_k}a_{i_kj},\ \ j=1,2,\dots,n,\ l=1,2,\dots,m.$$

Consider the system of homogeneous linear inequalities

$$A_{i_k}x\le 0,\ \ i_k\in I_n. \tag{2.2}$$

We indeed need to introduce the following definition.

Definition 1. *The linear inequality*

$$A_l x\le 0 \tag{2.3}$$

is called the consequent linear inequality of the system (2.2) if and only if all the solutions of the system (2.2) satisfy the linear inequality (2.3).

We need the following well known result in theory of linear inequalities.

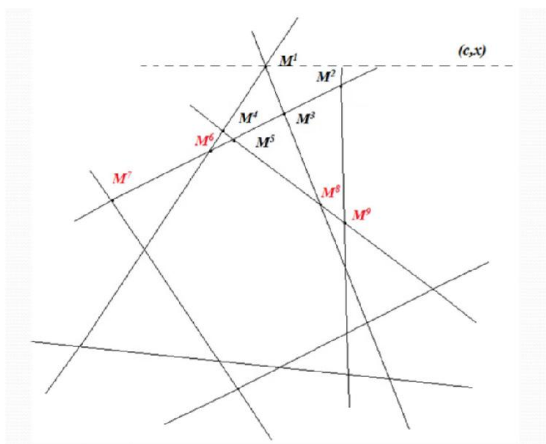
Theorem 2.1. *The linear inequality (2.3) is a consequent linear inequality of the system (2.2) if and only if*

$$A_l = \sum_{k=1}^{n} \lambda_{li_k} A_{i_k}, \qquad \lambda_{li_k} \geq 0, \qquad i_k \in I_n$$

Definition 2. *Let polyhedral cone M be defined by system*

$$\begin{aligned} &A_{i_1} x \leq b_{i_1}, \\ &A_{i_2} x \leq b_{i_2}, \\ &\ldots. \\ &A_{i_n} x \leq b_{i_n}, \end{aligned}$$

where $A_{i_1}, A_{i_2}, \ldots, A_{i_n}$ are linear independent. Then M is called a station cone if the vector c is a nonnegative linear combination of the vectors $A_{i_1}, A_{i_2}, \ldots, A_{i_n}$. The vertex $\overline{x}$ is called a station solution and the vectors $A_{i_1}, A_{i_2}, \ldots, A_{i_n}$ is called a basis of a station cone.



Therefore, geometrically it can be seen that all the station cones lie on one side of the objective function (c,x) at their vertices. In other words, the solutions of the system of linear inequalities that create the station cones satisfy the inequality $\langle c, x \rangle \leq \langle c, x^* \rangle$, whereas x* is the vertex of the station cones. This is equal to the fact that the inequality $\langle c, x \rangle \leq \langle c, x^* \rangle$ is the consequent inequality of the system of the linear inequalities, which formulate the station cone. This also means that the vector c is the nonnegative linear combination of the basic vectors of the station cone.

We have the following theorem

Theorem2.2. *If the station solution $\overline{x}$ satisfies all the constraints of the problem (2.1) then $\overline{x}$ is an optimal solution.*

3. Selecting the leaving vector

Let $A_{i_1}, A_{i_2}, \ldots, A_{i_n}$ be the basis of the station cone and

$$c = \sum_{k=1}^{n} \lambda_{k0} A_{i_k},$$

$$A_j = \sum_{k=1}^{n} \lambda_{kj} A_{i_k}, \quad j = 1, 2, \ldots m.$$

Then from definion 2.1 follows that

$$\lambda_{ko} \geq 0, \quad \forall k = 1, 2, \ldots n.$$

From now on we assume that all λ_{ko} are strictly positive, i.e.

$$\lambda_{k0} > 0, \ k = 1, 2, \ldots, n.$$

It is obvious that $\lambda_{k0} > 0, \ k = 1, 2, \ldots, n; \ \lambda_{k0} = 0, \ k = n+1, \ldots, m$ is a basis solution of the dual problem of (2.1):

$$\begin{aligned} &\min \ \langle b, \lambda \rangle \\ &A^T \lambda \geq c^T \qquad\qquad (3.1) \\ &\lambda \geq 0, \end{aligned}$$

where $\lambda \in R^m$. The assumption $\lambda_{k0} > 0, \ k = 1, 2, \ldots, n$ means that the dual problem (3.1) is nondegenerated.

Similarly to the simplex table, we establish the table of the coefficients λ_{kj} in the following way:

No	Basis	c	A_1	A_2	...	A_s	....	A_m
1	A_{i_1}	λ_{10}	λ_{11}	λ_{12}	....	λ_{1s}	....	λ_{1m}
2	A_{i_2}	λ_{20}	λ_{21}	λ_{22}	....	λ_{2s}	....	λ_{2m}
...	...	...	...	...	....	...	...	...
r	A_{i_r}	λ_{r0}	λ_{r1}	λ_{r2}	....	λ_{rs}	....	λ_{rm}
		...	...	...	...	...	...	...
n	A_{i_n}	λ_{n0}	λ_{n1}	λ_{n2}		λ_{ns}	....	λ_{nm}

Suppose A_{i_r} is the leaving vector and A_s is the entering vector. New coefficients λ'_{ko} of vector c can be calculated by the following formula

$$\lambda'_{r0} = \frac{\lambda_{r0}}{\lambda_{rs}},$$

$$\lambda'_{k0} = \lambda_{k0} - \frac{\lambda_{r0}}{\lambda_{rs}}\lambda_{ks}, \quad k \neq r, \tag{3.2}$$

$$\frac{\lambda_{k0}}{\lambda_{ks}} \geq \frac{\lambda_{r0}}{\lambda_{rs}}, \quad \lambda_{rs} > 0, \quad \lambda_{ks} > 0.$$

From

$\lambda_{ko} > 0,\ \lambda'_{ko} > 0, \quad \forall k = 1, 2, \ldots n$

we follow that

$$\lambda'_{r0} = \frac{\lambda_{r0}}{\lambda_{rs}} > 0,$$

$$\lambda'_{k0} = \lambda_{k0} - \frac{\lambda_{r0}}{\lambda_{rs}}\lambda_{ks} > 0,\ k \neq r.$$

$$\frac{\lambda_{k0}}{\lambda_{ks}} \geq \frac{\lambda_{r0}}{\lambda_{rs}}, \quad \lambda_{rs} > 0, \quad \lambda_{ks} > 0.$$

Therefore

$$\frac{\lambda_{r0}}{\lambda_{rs}} = \min_k \frac{\lambda_{k0}}{\lambda_{ks}}, \quad \lambda_{ks} > 0,\ \ \lambda_{rs} > 0\ . \tag{3.3}$$

The formulas (3.2), (3.3) guarantee that $A_{i_1}, A_{i_2}, \ldots, A_{i_{r-1}}, A_s, A_{i_{r+1}}, A_{i_n}$ are the basis of the station cone. So we have proved the following

Theoerem 2.3. *Let* $A_{i_1}, A_{i_2}, \ldots, A_{i_n}$ *be the basis of the station cone. Suppose we replaced* A_{i_r} *by* A_s. *Then* $A_{i_1}, \ldots, A_{i_{r-1}}, A_s, A_{i_{r+1}}, \ldots, A_{i_n}$ *is the basis of the station cone if the leaving vector* A_{i_r} *was choosen by condition*

$$\frac{\lambda_{r0}}{\lambda_{rs}} = \min_k \frac{\lambda_{k0}}{\lambda_{ks}}, \ \ \lambda_{ks} > 0,\ \ \lambda_{rs} > 0. \tag{3.4}$$

Now we have to show that formula (3.4) is hold.

Theorem 2.4. *Among the coefficients* λ_{ks} *, k = 1, 2,..., n at least one* λ_{rs} *exists such that* $\lambda_{rs} > 0$.

4. Selecting the entering vector

The idea of our algorithm is moving from one vertex x^k of a station cone M^k to another vertex $x^{\{k+1\}}$ of another station cone $M^{\{k+1\}}$ with a better value of the objective function. The movement depends on the cutting hyperplane $A_S x = b_S$ which will be defined by the intersection of the feasible polytope P and the segment connecting the vertex x^k of the station cone M^k and the given interior point $O \in P$. The movement stops when the vertex x^k of the station cone M^k becomes a feasible point. The number of iterations will depend on the method of selecting the cutting hyperplane. We will illustrate the idea and the effectiveness of our algorithm by considering the following examples.

Let us approximate the equator of the earth by a polygon with the edge of 1 meter long. Then this polygon has 40 millions edges and 40 millions vertices. Suppose we have to find the maximum of a linear function $cx_1 + cx_2$ over this polygon. Although this is only a 2-dimensional problem, but $m = 40.10^6$ is a huge number of constraints which will cause a tremendous difficulty.\\

On figure 1, let A denote an optimal point, B^1 denote the starting point. Suppose the distance between B^1 and A is 5 million meters. Then the simplex method will produce an optimal solution after 5 million iterations.

Let M^1be a station cone defined by 2 constraints containing points B^1 and D^1, where D^1 is on the other side of A with a distance, for examples, 4 million meters to A (see fig.1).

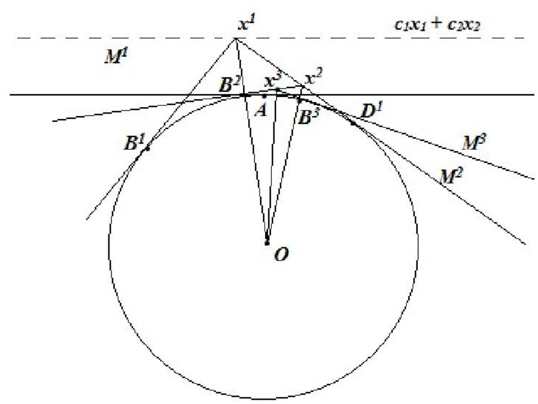


Fig.1

We denote by x^1 the vertex of M^1. Since M^1 is a station cone, it is clear that

$$cx^1 \geq cx, \forall x \in M^1.$$

The station cone M^1 will be our starting cone. Starting our algorithm with the operation of connecting x^1 with O, where O is the center of the equator. The segment $[x^1, O]$ will intersect with the boundary of P at B^2. Replacing the constraint containing B^1 by the constraint containing B^2 we have a new cone M^2. Repeat the above procedure with M^2 and we have M^3, etc. (see figure 1). The replacement of one constraint by another has to follow the restriction that the new generating cone is a station cone. We note that at each iteration, the distance between two points B^k and D^k defined by two edges of the station cone M^k is reduced by approximately 2 times in comparison with the previous iteration. Therefore the number of the iterations T can be estimated by the following bound

$$T \approx log_2 \frac{m}{2}$$

For our example with $m = 40$ million the formula (4.1) gives

$$T \approx log_2 \frac{m}{2} = log_2 2.10^7 < 25$$

The above example shows that the simplex method will produce an optimal solution after not more than $\frac{m}{2}$ iterations. But with $m = 40$ million constraints, $\frac{m}{2} = 20$ million is a huge number requiring a tremendous computational work. Our algorithm can produce an optimal solution after around 25 iterations.

We now proceed to find an initial station cone. We can find an initial station cone *M* by solving the following system

$$A^T\lambda = c^T, \quad \lambda \geq 0, \tag{4.2}$$

where $\lambda \in \mathbb{R}^m$. We can suppose $c^T \geq 0$ because, if some coefficient of c^T is negative then we multiply both sides of the corresponding equation with -1. To find a solution of (4.2), we solve the following big - M problem

$$\begin{aligned} &\min\{M_1y_1 + M_2y_2 + \ldots + M_ny_n\} \\ &A^T\lambda + Ey = c^T, \\ &\lambda \geq 0,\ y \geq 0, \end{aligned} \tag{4.3}$$

where, $\lambda \in \mathbb{R}^m$, $y \in \mathbb{R}^n$ and *E* is the unit matrix of ($n \times n$)and $M_1, M_2, \ldots, M_n$ are significantly large positive numbers. The problem (4.3) has an optimal solution $\lambda^* \geq 0,\ y^* = 0.$ And λ^* is a solution of (4.2).

We also assume that an strict interior feasible solution *O* of (2.1) is available. If such an initial point is not available then we modify the problem using the uasual big – M auggmentation [11] as follows:

$$\begin{aligned} &\max\ \{\langle c, x\rangle - Mx_{n+1}\} \\ &Ax - ex_{n+1} \leq b, \\ &x, x_{n+1} \geq 0. \end{aligned} \tag{4.4}$$

where $e = (1,1,\ldots,1)^T \in \mathbb{R}^m$ and *M* is a significantly large positive number. Let $x_{n+1}^0 > \max\{0, -b_1, -b_2, \ldots, -b_m\}$. Then $(0,\ldots,0,x_{n+1}^0)^T$ is a strict interior feasible solution of (4.4) which is in the same form as (2.1).

Let *O* be a strict interior point of *P*. Denoted by $O^i, i = 1, 2, \ldots, n$ the projections of O onto n facets of the station cone M^k. Let $H_i, i = 1, 2, \ldots, n$ be the intersection points of the boundary of

P and the segments $O, O^i, i = 1, 2, \ldots, n$. Then the new point O^* will be calculated by the following formula

$$O^* = \frac{1}{n+1}\left(\sum_{i=1}^{n} H_i + O\right) \tag{4.5}$$

Theorem 2.5. *Let x^k be a vertex of M^k at step k. Suppose x^k is a unique optimal solution of $\langle c, x\rangle$, $\forall x \in M^k$. Then*

$$\langle c, x^{k+1}\rangle < \langle c, x^k\rangle.$$

5. Algorithms

After the above discussion, we now proceed to formulate the following algorithm.

Algorithm 1

1. Initialization

Determine the starting station cone M. Calculate the point O^* by formula (4.5).

Let $M^k = M;\ O = O^*$.

2. Step ($k = 1, 2, \dots$)

If the vertex x^k of the station cone M^k is a feasible point of P, then x^k is an optimal solution. In the contrary case, select the inequality $A_s x \le b_s$ for entering the station cone and define the inequality $A_{i_r} x \le b_{i_r}$ for leaving the station cone. Determine the new station cone $M^{\{k+1\}}$ with the vertex $x^{\{k+1\}}$.

Go to next step $k = k + 1$.

With the assumption that the dual problem (3.1) of (2.1) is nondegenerated, we hence have the following

Theorem 2.6

The above algorithm produces an optimal solution after a finite number of iterations.

Proof. Follows from the theorems 2.3,2.4,2.5.

It is obvious that the calculation of the interior point O^* by formula (4.5) may require additional computational work which influences on the efficiency of the above algorithm. Normally we suggest using the point O instead of O^* if O is positioned quite distantly seperated from the facets of P. We also note that if O is near the optimal solution of (2.1) then the cutting hyperplane can most probably be one of the facets which formulate the optimal solution. Taking this advantage, we can have

$$O = \frac{1}{n} O + \frac{n-1}{n} z^k \text{ where } A_s z^k = b_s \tag{5.1}$$

It is clear that point O in the formula (5.1) is a strict interior point which is moving in the direction towards the optimal point step by step. Therefore we suggest the following

Algorithm2

1. Initialization

Determine the starting station cone M. Find the point O.

Let $M^k = M;\ O = O$

2. Step k ($k = 1, 2, \dots$)

If the vertex x^k of the station cone M^k is a feasible point of P, then x^k is an optimal solution. In the contrary case, select the inequality $A_s x \le b_s$ for entering the station cone and define the inequality $A_{i_r} x \le b_{i_r}$ for leaving the station cone.

Let $O = \frac{1}{2}(O + z^k)$ where $A_s z^k = b_s$

Determine the new station cone $M^{\{k+1\}}$ with the vertex $x^{\{k+1\}}$.

Go to next step $k = k + 1$.

6. Computational experiences

The proposed algorithm1 has been tested, using MatLab, on a set of randomly generated linear problems [13] of the form

$$\begin{cases} \max \ \langle c, x \rangle \\ \quad Ax \le b, \end{cases} \tag{7.1}$$

where $c = (1, 1, \dots, 1) \in R^n$, A is the full matrix of ($n \times m$) with a_{ij} is randomly generated from the interval [0,1), the vector b has been choosen such that the hyperplanes $\langle A_i, x \rangle = b_i,\ i = 1, \dots, m$ are tangent to the sphere (0,1) with center at origin and radius $r = 1$. To ensure that (7.1) has a finite optimal solution we add the contraints

$$x_i \leq 1, \quad i = 1, 2, \ldots, n. \tag{7.2}$$

The optimal solution and objective function value of ((7.1)-(7.2)) have been retested by simplex algorithm from MatLab.

Function **Data01. m** randomly generates the input data for the problems and stores the matrix A and, vector b in the data base form **Dat01. mat**. Function **Alg01. m** solves the problem by a new proposed algorithm and function **Simplex01. m** itself is the simplex algorithm from the optimization toolbox of MatLab.

Table 1. $m = 2n$.

n	m	problem	Iterations		Ratio (SIMPLEX/INEX)
			PHASE II SIMPLEX	INTERIOR EXTERIOR APPROACH	
10	20	1	31	15	
		2	31	17	
		3	38	15	
		Average	**33**	**16**	**2.06**
20	40	1	126	53	
		2	134	44	
		3	157	51	
		Average	**139**	**49**	**2.84**
40	80	1	631	178	
		2	623	206	
		3	656	163	
		Average	**637**	**182**	**3.5**
60	120	1	1320	290	
		2	1669	328	
		3	1406	340	
		Average	**1465**	**319**	**4.59**

7. Conclusions

The above proposed method can be viewed as a variation of simplex algorithm in combination with the interior approach. The above tested examples show that the number of iterations of the interior exterior approach is significantly smaller than that of the second phase of the simplex method. Additionally, when the number of variables and constraints of the problem increase, the number of iterations of the interior exterior approach increase in a slower manner than that of the simplex method. The tested experiements show that with a fixed number of variables, the number of iterations produced by the interior exterior approach seems to be impacted at a lower level by an increase in the number of the constraints in comparison with the simplex method. In order to gain a more precise conclusion on the effectiveness of the above proposed algorithm, there is a need to carry out computational experiments on a larger scale together with a bigger number of variables and constraints.

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