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Finite-time stability of singular nonlinear switched time-delay systems: A singular value decomposition approach

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Abstract

In this paper, a constructive geometric design of switching laws is proposed for the finite-time stability of singular nonlinear switched systems subjected to delay and disturbance. The state-dependent switching law is constructed based on the construction of a partition of the stability state regions in convex cones such that each system mode is activated in one particular conic zone. Using the state-space singular value decomposition approach, new delay-dependent sufficient conditions for the finite-time stability of the system are presented in terms of linear matrix inequalities (LMIs). The obtained results are applied to uncertain linear singular switched systems with delay. Numerical examples are given to illustrate the effectiveness of the proposed method.

1. Introduction

Switched systems belong to an important class of hybrid systems arise in many practical processes that cannot be described by exclusively continuous or exclusively discrete models,

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such as manufacturing, communication networks, automotive engineering control, chemical processes [1,2]. Many effective methods have been proposed to studying stability and control of switched systems, such as the Lyapunov function approach, the LMI approach, and average dwell-time scheme [3–5]. On the other hand, stability and control problems of singular systems have been extensively studied due to the fact that the singular system better describes physical systems than state-space systems [6–9]. Recently stability and stabilization of singular fuzzy systems [10] have been addressed in [10] by using non-quadratic Lyapunov functions and S-procedure. For switched singular systems without delay, the stability analysis was discussed in [11] by using piecewise Lyapunov function method and switching law satisfying an average dwell-time constraint. For the systems with delay, because of the combination between the switching and the time-delay and due to the algebraic constraints in singular models, the stability analysis of such systems is much more complicated than that of singular systems without delays. Based on the average dwell-time approach, the authors of Xing and Min [12] and Krishnasamy and Balasubramaniam [13] proposed some sufficient conditions for robust exponential admissibility of singular linear switched systems and the authors of Zamani et al. [14] extended the existing results on exponential stability of singular nonlinear switched systems with time-varying delay. It should be noticed that most of the mentioned papers are focused on the asymptotic stability. In many practical applications, the main concern is the behavior of the system over a fixed finite time interval. In these cases, finite-time stability could be used, which focuses its attention on the transient behavior over a finite time interval rather than on the asymptotic behavior of a system response [15,16]. This different concept of finite-time stability requires convergence of the system trajectories to an equilibrium state in finite-time, which does not require the specification of any bounding region. The finite-time stability not only deals with systems whose operation is limited to a fixed finite interval of time, but also requires prescribed bounds on system variables. In the analysis of finite-time stability, the assumption of system asymptotic stability is unnecessary, i.e. the unstable system can be finite-time stable.

To the best of our knowledge, up to the present days there have been few results on finite-time stability for singular nonlinear switched systems with delay reported in the literature. Existing attempts for finite-time stability analysis of singular linear switched systems are mere extensions of the Lyapunov methods for individual linear singular singular systems. The papers [17–19] have studied the finite-time stability problem for switched linear systems, but singularity case is not considered. In [20,21], the authors have investigated the finite-time stabilization problem for switched linear singular systems, but the time delay is not considered. Finite-time stability for singular linear time-delay systems with time-varying exogenous disturbance was studied in [22], but the switching case is not considered. Moreover, the majority of the previous works treated the stability for switched linear systems under arbitrary switching laws or switching signals specified by the average dwell-time.

In this paper, we consider finite-time stability for a class of nonlinear singular switched time-delay systems. More precisely, a constructive geometric design of state-dependent switching laws is proposed for finite-time stability of the system based on the state-space singular value decomposition approach. The main contributions of this paper can be highlighted in three points: (i) a constructive geometric design of the switching laws is proposed using the construction of a partition of the state space in convex cones such that each system mode is active in one particular conic zone and each subregion is defined to make particular quadratic form negative; (ii) a set of novel delay-dependent finite-time stability conditions is established in terms of LMIs, which can be determined by utilizing MATLABs LMI Control Toolbox;

(iii) an application to finite-time stability is given for a class of uncertain linear singular switched time-delay systems.

The paper is organized in the following way. Section 2 presents some notations, definitions and preliminary propositions which will be used in next sections. In Section 3, we address the switching design problem, namely some sufficient conditions for constructing state-dependent switching law guaranteeing the finite-time stability of the system in term of LMIs. An application to uncertain singular linear switched time-delay systems and illustrative examples are given in this section. Some conclusions are drawn in Section 4.

2. Problem description and preliminaries

The following notations will be used throughout this paper. R^+ denotes the set of all real positive numbers; R^n denotes the $n-$ dimensional space with its norm $\|x\| = \sqrt{x^\top x}$; $R^{n \times r}$ denotes the space of all $(n \times r)$-matrices; the notation $i = \overline{1, p}$ means $i = 1, 2, \dots, p$; $A^\top$ denotes the transpose of A; I_r and 0_r denote the identity matrix and zero matrix in $R^{r \times r}$, respectively; $\lambda(A)$ denotes the set of all eigenvalues of A; $[a_{ij}]_{i,j=\overline{1,k}}$ denotes the $(k \times k)$-matrix with entries a_{ij}; $\lambda_{max}(A) = \max\{\mathrm{Re}\lambda : \lambda \in \lambda(A)\}$; $\lambda_{min}(A) = \min\{\mathrm{Re}\lambda : \lambda \in \lambda(A)\}$; $\lambda_A = \lambda_{max}(A^\top A)$; the matrix norm $||A|| = \sqrt{\lambda_{max}(A^\top A)}$; $C([a, b], R^n)$ denotes the set of all R^n-valued continuous functions on $[a, b]$; matrix A is semi-positive definite ($A \geq 0$) if $x^\top Ax \geq 0$, for all $x \in R^n$; A is positive definite ($A > 0$) if $x^\top Ax > 0$ for all $x \neq 0$; $A \geq B$ means $A - B \geq 0$; the segment of the trajectory $x(t)$ is denoted by $x_t = \{x(t+s) : s \in [-h, 0]\}$ with its norm $||x_t|| = \sup_{s\in[-h,0]} ||x(t+s)||$.

Consider a singular nonlinear switched system with delay and disturbance of the form:

$$\begin{cases} E\dot{x}(t) = & A_\sigma x(t) + D_\sigma x(t-h) + B_\sigma \omega(t) + f_\sigma(t, x(t), x(t-h), \omega(t)), \ t \geq 0, \\ x(\theta) = & \varphi(\theta), \ \theta \in [-h, 0], \end{cases} \tag{1}$$

where the function $\sigma : R^n \to \{1, \dots, p\}$ is a switching rule depending on the system state at each time and takes its values in the finite set of modes $\{1, \dots, p\}$; the system matrices $(A_\sigma, D_\sigma, B_\sigma)$ take values in the finite set of (A_l, D_l, B_l), $l \in \overline{1, p}$, where A_l, $D_l \in R^{n \times n}$, $B_l \in R^{n \times q}$ are given constant matrices; the matrix $E \in R^{n \times n}$ is singular and rank $E = r < n$; the initial condition $\varphi \in C([-h, 0], R^n)$ and the exogenous disturbance $\omega(t)$ is continuous satisfying the condition:

$$\exists d > 0 : \quad \omega(t)^\top \omega(t) \leq d, \quad \forall t \geq 0; \tag{2}$$

the nonlinear function $f_l(\cdot)$ satisfies the condition:

$$\exists a_l, b_l, m_l > 0 : \quad \|f_l(t, x, x_h, \omega)\| \leq a_l\|x\| + b_l\|x_h\| + m_l\|\omega\|, \tag{3}$$

for all $(t, x, x_h, \omega) \in R^+ \times R^n \times R^n \times R^q$.

Corresponding to the switching law $\sigma(x(t))$, we assume that the system is activated by the lth switching mode, which means that $\sigma(x(t)) = l$.

Definition 1. For the switching law $\sigma(.)$, the system (1) is said to be (i) regular if the polynomial $det(sE - A_l)$ is not identically zero for each $\sigma(x(t)) = l$; (ii) impulse-free if the $\deg(\det(sE - A_l)) = \text{rank } E$ for each $\sigma(x(t)) = l$.

Definition 2. For given positive numbers T, c_1, c_2, and a symmetric positive definite matrix $Q \in R^{n \times n}$, the system (1) is finite-time stable w.r.t. (c_1, c_2, T, Q) under switching law $\sigma(\cdot)$

if it is regular, impulse-free and every solution $x_\sigma(t, \varphi)$ of the system satisfies the condition:

$$\sup_{s\in[-h,0]} \{\varphi(s)^\top Q\varphi(s)\} \le c_1 \ \Rightarrow\ x_\sigma(t,\varphi)^\top Q x_\sigma(t,\varphi) \le c_2,\ \forall t \in [0,T],$$

for all disturbances $\omega(\cdot)$ satisfying Eq. (2).

The problem to be addressed in this paper is to identify a class of switching laws $\sigma(.)$ such that the system is finite-time stable under the switching $\sigma(.)$. We will construct the switching laws for the system (1) based on the construction of a partition of the state space in convex cones such that each system mode is activated in one particular conic zone and each subregion is defined to make particular quadratic form negative.

Definition 3. The system of matrices $\{L_i\}_{i=1}^p$ is strictly complete if for every $x \in R^n\backslash\{0\}$ there is $i \in \{1, 2, \dots, p\}$ such that $x^\top L_i x < 0$.

It is easy to see that system $\{L_i\}$ is strictly complete if and only if

$$\bigcup_{i=1}^p \Omega_i = R^n \setminus \{0\},\ \text{where}\ \ \Omega_i = \{x \in R^n :\ x^\top L_i x < 0\},\ i = \overline{1,p}.$$

Proposition 1. *[23] System $\{L_i\}_{i=1}^p$ is strictly complete if there exist numbers $\xi_i \ge 0,\ i = \overline{1,p},\ \sum_{i=1}^p \xi_i > 0$, such that $\sum_{i=1}^p \xi_i L_i < 0$.*

Proposition 2 (Schur Complement Lemma [24])**.** *Given constant matrices X, Y, Z, where $Y = Y^\top > 0$, we have*

$$X + Z^\top Y^{-1} Z < 0 \Longleftrightarrow \begin{bmatrix} X & Z^\top \\ Z & -Y \end{bmatrix} < 0.$$

3. Main results

The purpose of this section is to study finite-time stability of singular linear switched system (1). We first establish delay-dependent conditions to check the regularity and impulse-free of the systems based on singular value decomposition method. Then, we prove the finite-time stability based on the Lyapunov-like function method. Consider the system (1), where rank $E = r < n$. Then there are two nonsingular matrices M, G such that

$$MEG = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}.$$

Let us set

$$MA_lG = \begin{bmatrix} A_{11}^l & A_{12}^l \\ A_{21}^l & A_{22}^l \end{bmatrix},\quad MD_lG = \begin{bmatrix} D_{11}^l & D_{12}^l \\ D_{21}^l & D_{22}^l \end{bmatrix},$$

$$MB_l = \begin{bmatrix} B_1^l \\ B_2^l \end{bmatrix},\quad Mf_l(\cdot) = \begin{bmatrix} f_1^l(\cdot) \\ f_2^l(\cdot) \end{bmatrix}.$$

Under the state transformation $y(t) = G^{-1}x(t), y(t)^\top = (y_1(t)^\top,\ y_2(t)^\top),\ y_1(t) \in R^r,\ y_2(t) \in R^{n-r},\ \sigma(.) = l$, the system (1) takes the following form

$$\begin{cases} \dot{y}_1(t) = A^l_{11} y_1(t) + A^l_{12} y_2(t) + D^l_{11} y_1(t-h) + D^l_{12} y_2(t-h) + B^l_1 \omega(t) + f^l_1(\cdot), \\ 0 = A^l_{21} y_1(t) + A^l_{22} y_2(t) + D^l_{21} y_1(t-h) + D^l_{22} y_2(t-h) + B^l_2 \omega(t) + f^l_2(\cdot), \\ y(t) = G^{-1}\varphi(t), \ \ t \in [-h, 0]. \end{cases} \tag{4}$$

Before introducing the main result, the following notations of several matrix variables are defined for simplicity.

$$\overline{M} = \begin{bmatrix} 0 & 0 \\ 0 & I_{n-r} \end{bmatrix} M, \ G^\top PEG = \begin{bmatrix} P_1 & 0 \\ 0 & 0 \end{bmatrix}, \ a = \max_{l=\overline{1,p}} a_l, \ \ b = \max_{l=\overline{1,p}} b_l, \ \ m = \max_{l=\overline{1,p}} m_l,$$

$$\gamma_0 = 1 - a||G|| \max_{l=\overline{1,p}} \|[A^l_{22}]^{-1}\|, \ \gamma_1 = \max_{l=\overline{1,p}} \Big(\|[A^l_{22}]^{-1} B^l_2\| \sqrt{d} + m \|[A^l_{22}]^{-1}\| \sqrt{d} \Big),$$

$$\gamma_2 = \max_{l=\overline{1,p}} \Big(\|[A^l_{22}]^{-1} A^l_{21}\| + a||G||.||[A^l_{22}]^{-1}|| \Big), \ \gamma_3 = \max_{l=\overline{1,p}} \Big(\|[A^l_{22}]^{-1} D^l_{21}\| + b||G||.||[A^l_{22}]^{-1}|| \Big),$$

$$\gamma_4 = \max_{l=\overline{1,p}} \Big(\|[A^l_{22}]^{-1} D^l_{22}\| + b||G||.||[A^l_{22}]^{-1}|| \Big), \ \gamma_5 = \sqrt{\frac{\alpha_2 c_1 + (2+2m)dT}{\alpha_1}}, \ \gamma_6 = \sum_{i=0}^{[\frac{T}{h}]} \alpha_5^i,$$

$$\gamma_7 = \max_{i=0,1,2,\ldots,[\frac{T}{h}]} \{\alpha_5^{i+1} \sqrt{\alpha_3 c_1}\}, \ \gamma_8 = \lambda_{max}(G^\top Q G), \ \alpha_1 = \lambda_{min}(P_1),$$

$$\alpha_2 = \frac{\lambda_{max}(PE)}{\lambda_{min}(Q)} + h \frac{\lambda_{max}(U)}{\lambda_{min}(Q)},$$

$$\alpha_3 = \frac{\lambda_{max}([G^{-1}]^\top [G^{-1}])}{\lambda_{min}(Q)}, \ \alpha_4 = \frac{\gamma_1 + \gamma_3 \sqrt{\alpha_3 c_1}}{\gamma_0} + \frac{\gamma_2\gamma_5 + \gamma_3\gamma_5}{\gamma_0} e^{0.5\beta T}, \ \alpha_5 = \frac{\gamma_4}{\gamma_0},$$

$$\alpha_6 = (\gamma_7 + \gamma_6 \frac{\gamma_1 + \gamma_3\sqrt{\alpha_3 c_1}}{\gamma_0} + \gamma_6 \frac{\gamma_2\gamma_5 + \gamma_3\gamma_5}{\gamma_0} e^{0.5\beta T})^2, \ \alpha_9 = \gamma_8 \gamma_5^2 + \gamma_8 \gamma_6^2 \Big(\frac{\gamma_2\gamma_5 + \gamma_3\gamma_5}{\gamma_0} \Big)^2,$$

$$\alpha_7 = 2\gamma_8 \Big(\frac{\gamma_2\gamma_5 + \gamma_3\gamma_5}{\gamma_0} \Big) \Big(\frac{\gamma_1 + \gamma_3\sqrt{\alpha_3 c_1}}{\gamma_0} \gamma_6^2 + \gamma_6\gamma_7 \Big), \ \alpha_8 = \gamma_8 \Big(\gamma_7 + \gamma_6 \frac{\gamma_1 + \gamma_3 \sqrt{\alpha_3 c_1}}{\gamma_0} \Big)^2,$$

$$H^l_{11} = 0.5PA_l + 0.5A_l^\top P^\top + 0.5 S_l \overline{M} A_l + 0.5 A_l^\top \overline{M}^\top S_l^\top, \ H^l_{12} = PD_l + S_l \overline{M} D_l,$$

$$H^l_{22} = -U + 2b_l I_n,$$

$$H^l_{33} = -I_q, \ H^l_{13} = S_l \overline{M} B_l, \ H^l_{44} = -I, \ \ H^l_{14} = \sqrt{a_l + b_l + m_l} \ S_l \overline{M}, \ H^l_{55} = -I_q, \ \ H^l_{15} = PB_l,$$

$$H^l_{66} = -I, \ \ H^l_{16} = \sqrt{a_l + b_l + m_l} \ P, \ H^l_{ij} = H^l_{ji}, \ i, j = \overline{1,6},$$

$$L_l = 0.5PA_l + 0.5A_l^\top P^\top + 0.5 S_l \overline{M} A_l + 0.5 A_l^\top \overline{M}^\top S_l^\top + U + 2a_l I_n,$$

$$\Omega_l = \{x \in R^n : \ x^\top L_l x < 0\}, \ \ l = \overline{1,p}, \ \ \overline{\Omega}_1 = \Omega_1 \cup \{0\}, \ \ \overline{\Omega}_l = \Omega_l \setminus \bigcup_{k=1}^{l-1} \overline{\Omega}_k, \ \ l = 2, 3, \ldots, p.$$

Theorem 1. *For given positive numbers T, c_1, c_2, and a symmetric positive definite matrix $Q \in R^{n \times n}$, the system (1) is finite-time stable w.r.t. (c_1, c_2, T, Q) if there exist a symmetric positive definite matrix $U \in R^{n \times n}$, a nonsingular matrix $P \in R^{n \times n}$, any matrices $S_l \in R^{n\times n}$, $l \in \overline{1,p}$, scalars $\xi_l \geq 0$, $l = \overline{1,p}$, $\sum_{l=1}^p \xi_l > 0$, and a number $\beta > 0$ such that the following conditions hold:*

$$PE = E^\top P^\top \geq 0, \tag{5}$$

$$[H^l_{ij}]_{i,j=\overline{1,6}} < 0, \quad l = \overline{1,p}, \tag{6}$$

$$\sum_{l=1}^{p} \xi_l L_l < 0, \tag{7}$$

$$1 - a||G|| \max_{l=\overline{1,p}} \|[A^l_{22}]^{-1}\| > 0, \tag{8}$$

$$\begin{bmatrix} \alpha_7 e^{0.5\beta T} + \alpha_8 - c_2 & e^{0.5\beta T} \\ e^{0.5\beta T} & -\dfrac{1}{\alpha_9} \end{bmatrix} < 0. \tag{9}$$

The switching rule is chosen as $\sigma(x(t)) = l$ *whenever* $x(t) \in \overline{\Omega}_l$.

Proof. The proof is divided into two steps. The first step is to prove the regularity and the impulse-free of the singular system (1). The second step will focus on getting conditions for design the state-dependent switching laws for finite-time stability by using Lyapunov-like function method and LMI technique.

Step 1. The regularity and impulse-free of the system.

Let us set

$$G^\top = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}, \ S_l = \begin{bmatrix} S^l_{11} & S^l_{12} \\ S^l_{21} & S^l_{22} \end{bmatrix}, \ G^\top P M^{-1} = \begin{bmatrix} P_1 & P_{12} \\ P_{21} & P_{22} \end{bmatrix}.$$

From the condition of Eq. (5), $PE = E^\top P^\top \geq 0$, it is easily seen that

$$G^\top PEG = G^\top PM^{-1}MEG = G^\top PM^{-1}\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} P_1 & 0 \\ P_{21} & 0 \end{bmatrix} \geq 0,$$

$$G^\top E^\top P^\top G = \begin{bmatrix} P_1^\top & P_{21}^\top \\ 0 & 0 \end{bmatrix} \geq 0,$$

and hence

$$P_{21} = 0, \ \ P_1 = P_1^\top \geq 0, \ \ G^\top PEG = \begin{bmatrix} P_1 & 0 \\ 0 & 0 \end{bmatrix}. \tag{10}$$

Since matrix P is nonsingular, it follows the non-singularity of $G^\top PM^{-1} = [\begin{smallmatrix} P_1 & P_{12} \\ 0 & P_{22} \end{smallmatrix}]$ such that from Eq. (10) it follows that $det(P_1) \neq 0$, and hence $P_1 > 0$. Next, note that the LMI (6) implies the following inequality

$$H^l_{11} = 0.5PA_l + 0.5A_l^\top P^\top + 0.5S_l\overline{M}A_l + 0.5A_l^\top \overline{M}^\top S_l^\top < 0,$$

consequently,

$$G^\top[PA_l + A_l^\top P^\top + S_l\overline{M}A_l + A_l^\top \overline{M}^\top S_l^\top]G < 0. \tag{11}$$

On the other hand, we rewrite the expression $G^\top S_l \overline{M} A_l G$ and $G^\top PA_l G$ as follows

$$\begin{aligned} G^\top PA_lG = G^\top PM^{-1}MA_lG &= \begin{bmatrix} P_1 & P_{12} \\ 0 & P_{22} \end{bmatrix}\begin{bmatrix} A^l_{11} & A^l_{12} \\ A^l_{21} & A^l_{22} \end{bmatrix} \\ &= \begin{bmatrix} P_1A^l_{11} + P_{12}A^l_{21} & P_1A^l_{12} + P_{12}A^l_{22} \\ P_{22}A^l_{21} & P_{22}A^l_{22} \end{bmatrix}, \end{aligned}$$

$$\begin{aligned}G^\top S_l\overline{M}A_lG &= G^\top\begin{bmatrix}S^l_{11} & S^l_{12}\\ S^l_{21} & S^l_{22}\end{bmatrix}\begin{bmatrix}0 & 0\\ 0 & I_{n-r}\end{bmatrix}MA_lG\\ &= \begin{bmatrix}G_{11} & G_{12}\\ G_{21} & G_{22}\end{bmatrix}\begin{bmatrix}S^l_{11} & S^l_{12}\\ S^l_{21} & S^l_{22}\end{bmatrix}\begin{bmatrix}0 & 0\\ 0 & I_{n-r}\end{bmatrix}\begin{bmatrix}A^l_{11} & A^l_{12}\\ A^l_{21} & A^l_{22}\end{bmatrix}\\ &= \begin{bmatrix}G_{11}S^l_{12}A^l_{21}+G_{12}S^l_{22}A^l_{21} & G_{11}S^l_{12}A^l_{22}+G_{12}S^l_{22}A^l_{22}\\ G_{21}S^l_{12}A^l_{21}+G_{22}S^l_{22}A^l_{21} & G_{21}S^l_{12}A^l_{22}+G_{22}S^l_{22}A^l_{22}\end{bmatrix}.\end{aligned}$$

Therefore, taking Eq. (11) into account we have

$$\Big[G_{21}S^l_{12}+G_{22}S^l_{22}+P_{22}\Big]A^l_{22}+[A^l_{22}]^\top\Big[G_{21}S^l_{12}+G_{22}S^l_{22}+P_{22}\Big]^\top<0,$$

which gives $det(A^l_{22})\neq 0$ and then the system is regular and impulse-free (see [8]).

Step 2. Finite-time stability.

Consider the following non-negative quadratic function:

$$V(t,x_t)=e^{\beta t}x(t)^\top PEx(t)+e^{\beta t}\int_{t-h}^{t}x(s)^\top Ux(s)ds.$$

Assuming that the system is activated by the lth switching mode, which means that $\sigma(x(t))=l$, we take the derivative of $V(t, x_t)$ in t along the solution of the system:

$$\begin{aligned}\dot V(t,x_t)=&\beta V(t,x_t)+e^{\beta t}x(t)^\top Ux(t)-e^{\beta t}x(t-h)^\top Ux(t-h)\\ &+e^{\beta t}2x(t)^\top P[A_lx(t)+D_lx((t-h)+B_l\omega(t)+f_l(t,x(t),x(t-h),\omega(t))].\end{aligned}\tag{12}$$

To estimate the derivative of $V(t, x_t)$, we need the following inequalities. Firstly, multiplying the both side of the following identity by $2e^{\beta t}x(t)^\top S_l\overline{M}$ from the left hand side

$$-E\dot x(t)+A_lx(t)+D_lx(t-h)+B_l\omega(t)+f_l(\cdot)=0,$$

and noting that $\overline{M}E=0$, we have

$$0=2e^{\beta t}x(t)^\top S_l\overline{M}[A_lx(t)+D_lx(t-h)+B_l\omega(t)+f_l(\cdot)].\tag{13}$$

Using Cauchy matrix inequality for the following estimations:

$$\begin{aligned}2x(t)^\top S_l\overline{M}B_l\omega(t)\leq&x(t)^\top S_l\overline{M}B_lB_l^\top\overline{M}^\top S_l^\top x(t)+\|\omega(t)\|^2,\\ 2x(t)^\top S_l\overline{M}f_l(\cdot)\leq&2||x(t)^\top S_l\overline{M}||||f_l(\cdot)||\\ \leq&2||x(t)^\top S_l\overline{M}||[a_l||x(t)||+b_l||x(t-h)||+m_l||\omega(t)||]\\ \leq&(a_l+b_l+m_l)||x(t)^\top S_l\overline{M}||^2+a_l||x(t)||^2+b_l||x(t-h)||^2+m_l||\omega(t)||^2,\\ 2x(t)^\top PB_l\omega(t)\leq&x(t)^\top PB_lB_l^\top P^\top x(t)+\|\omega(t)\|^2,\\ 2x(t)^\top Pf_l(\cdot)\leq&2||x(t)^\top P||||f_l(\cdot)||\\ \leq&2||x(t)^\top P||[a_l||x(t)||+b_l||x(t-h)||+m_l||\omega(t)||]\\ \leq&(a_l+b_l+m_l)||x(t)^\top P||^2+a_l||x(t)||^2+b_l||x(t-h)||^2+m_l||\omega(t)||^2,\end{aligned}$$

we obtain from Eqs. (12) and (13) that

$$\dot V(\cdot)-\beta V(\cdot)\leq e^{\beta t}(2+2m)||\omega(t)||^2+e^{\beta t}\xi(t)^\top C_l\xi(t)+e^{\beta t}x(t)^\top L_lx(t),\tag{14}$$

where $\xi(t)^\top=[x(t)^\top,\ x(t-h)^\top]$, and $C_l=[\begin{smallmatrix}C^l_{11} & C^l_{12}\\ C^l_{21} & C^l_{22}\end{smallmatrix}]$,

$$\begin{aligned}
C_{11}^l &= 0.5PA_l + 0.5A_l^\top P^\top + 0.5S_l\overline{M}A_l + 0.5A_l^\top\overline{M}^\top S_l^\top + S_l\overline{M}B_lB_l^\top\overline{M}^\top S_l^\top \\
&\quad + (a_l + b_l + m_l)S_l\overline{M}\ \overline{M}^\top S_l^\top + PB_lB_l^\top P^\top + (a_l + b_l + m_l)PP^\top, \\
C_{12}^l &= PD_l + S_l\overline{M}D_l,\ C_{22}^l = -U + 2b_lI_n, \\
L_l &= 0.5PA_l + 0.5A_l^\top P^\top + U + 0.5S_l\overline{M}A_l + 0.5A_l^\top\overline{M}^\top S_l^\top + 2a_lI_n.
\end{aligned}$$

Since the system of matrices $\{L_l:\ l = \overline{1,p}\}$ is strictly complete due to Eq. (7) and Proposition 1, we get

$$\bigcup_{l=1}^{p}\Omega_l = R^n \setminus \{0\},$$

and hence by constructing the sets $\overline{\Omega}_l$, we have

$$\bigcup_{l=1}^{p}\overline{\Omega}_l = R^n \quad \text{and} \quad \overline{\Omega}_{l_1} \cap \overline{\Omega}_{l_1} = \emptyset, \quad \text{for all} \quad l_1 \neq l_2.$$

Therefore, for $x(t) \in R^n$, there exists a unique $l \in \{1, 2, \ldots, p\}$ such that $x(t) \in \overline{\Omega}_l$ and

$$x(t)^\top L_l x(t) \leq 0. \tag{15}$$

Choosing the switching rule as $\sigma(x(t)) = l$ whenever $x(t) \in \overline{\Omega}_l$. Using the Schur complement lemma, Proposition 2, the condition (6) leads to $C_l < 0,\ l = \overline{1,p}$, and from the inequalities (14) and (15), it follows that

$$\dot V(t, x_t) - \beta V(t, x_t) \leq e^{\beta t}(2 + 2m)\omega(t)^\top\omega(t),\ \ \forall t \geq 0. \tag{16}$$

Multiplying the both sides of Eq. (16) by $e^{-\beta t}$ and noting that

$$\frac{d}{dt}(e^{-\beta t}V(t, x_t)) = e^{-\beta t}\dot V(t, x_t) - \beta e^{-\beta t}V(t, x_t),$$

we have

$$\frac{d}{dt}(e^{-\beta t}V(t, x_t)) \leq (2 + 2m)\omega(t)^\top\omega(t),\ \ \forall t \geq 0.$$

Integrating the above inequality from 0 to t, we obtain

$$\begin{aligned}
e^{-\beta t}V(t, x_t) - V(0, x_0) &\leq \int_0^t (2 + 2m)\omega(s)^\top\omega(s)ds \\
&\leq \int_0^t (2 + 2m)\omega(s)^\top\omega(s)ds \leq (2 + 2m)dT,
\end{aligned}$$

and hence

$$V(t, x_t) \leq [V(0, x_0) + (2 + 2m)dT]e^{\beta T},\ \ \forall t \in [0, T]. \tag{17}$$

Taking the condition (10) and the condition $P_1 > 0$ into account combining with the expression of $V(t,\ x_t)$, we obtain that

$$\begin{aligned}
V(t, x_t) &\geq x(t)^\top PEx(t) = y(t)^\top G^\top PEGy(t) = y_1(t)^\top P_1y_1(t) \\
&\geq \lambda_{min}(P_1)y_1(t)^\top y_1(t)) = \alpha_1y_1(t)^\top y_1(t).
\end{aligned} \tag{18}$$

On the other hand, since

$$\begin{aligned} V(0,x_0) &= x(0)^\top PEx(0) + \int_{-h}^{0} x(s)^\top Ux(s)ds \\ &\le \frac{\lambda_{max}(PE)}{\lambda_{min}(Q)} x(0)^\top Qx(0) + h\frac{\lambda_{max}(U)}{\lambda_{min}(Q)} \sup_{s\in[-h,0]} x(s)^\top Qx(s) \\ &\le \alpha_2 \sup_{s\in[-h,0]} \varphi(s)^\top Q\varphi(s) \le \alpha_2 c_1. \end{aligned} \tag{19}$$

we obtain from Eqs. (17)–(19) that

$$||y_1(t)|| \le \sqrt{\frac{1}{\alpha_1}[V(0,x_0) + (2+2m)dT]e^{\beta T}} \le \gamma_5 e^{0.5\beta T}, \ \ \forall t \in [0, T]. \tag{20}$$

Next, we estimate the second state $\|y_2(t\|$ as follows. Consider the second equation of (4)

$$y_2(t) = -[A_{22}^l]^{-1}\Big[A_{21}^l y_1(t) + D_{21}^l y_1(t-h) + D_{22}^l y_2(t-h) + B_2^l\omega(t) + f_2^l(\cdot)\Big].$$

Using the inequality (20), we have

$$\begin{aligned} \|y_2(t)\| \le& \|[A_{22}^l]^{-1}A_{21}^l\|\|y_1(t)\| + \|[A_{22}^l]^{-1}D_{21}^l\|\|y_1(t-h)\| \\ &+ \|[A_{22}^l]^{-1}D_{22}^l\|\|y_2(t-h)\| + \|[A_{22}^l]^{-1}B_2^l\|\|\omega(t)\| + \|[A_{22}^l]^{-1}\|\|f_2^l(\cdot)\| \\ \le& \|[A_{22}^l]^{-1}A_{21}^l\|\|y_1(t)\| + \|[A_{22}^l]^{-1}D_{21}^l\|\|y_1(t-h)\| \\ &+ \|[A_{22}^l]^{-1}D_{22}^l\|\|y_2(t-h)\| + \|[A_{22}^l]^{-1}B_2^l\|\|\omega(t)\| \\ &+ \|[A_{22}^l]^{-1}\|(a||G||[||y_1(t)|| + ||y_2(t)||] + b||G||[||y_1(t-h)|| \\ &+ ||y_2(t-h)||] + m\sqrt{d}) \\ \le& a||G|| \max_{l=\overline{1,p}} \|[A_{22}^l]^{-1}\|||y_2(t)|| + \Big(\|[A_{22}^l]^{-1}A_{21}^l\| + a||G||\|[A_{22}^l]^{-1}||\Big)\gamma_5 e^{0.5\beta T} \\ &+ \Big(\|[A_{22}^l]^{-1}D_{21}^l\| + b||G||\|[A_{22}^l]^{-1}||\Big)\big(\gamma_5 e^{0.5\beta T} + \sqrt{\alpha_3 c_1}\big) \\ &+ \|[A_{22}^l]^{-1}B_2^l\|\sqrt{d} + m\|A_{22}^l]^{-1}\|\sqrt{d} + \Big(\|[A_{22}^l]^{-1}D_{22}^l\| + b||G||\|[A_{22}^l]^{-1}\|\Big)\|y_2(t-h)\| \\ \le& a||G|| \max_{l=\overline{1,p}} \|[A_{22}^l]^{-1}\|||y_2(t)|| + \alpha_4\gamma_0 + \alpha_5\gamma_0\|y_2(t-h)\| \end{aligned} \tag{21}$$

because of the estimations of $\|f_2^l(\cdot)\|$ and $\|y_1(t-h)\|$ on [0, T] as

$$\begin{aligned} \|f_2^l(\cdot)\| &\le \|f_l(\cdot)\| \le a_l||x(t)|| + b_l||x(t-h)|| + m_l||\omega(t)|| \\ &\le a_l||Gy(t)|| + b_l||Gy(t-h)|| + m_l||\omega(t)|| \\ &\le a||G||\Big[||y_1(t)|| + ||y_2(t)||\Big] + b||G||\Big[||y_1(t-h)|| + ||y_2(t-h)||\Big] + m\sqrt{d}, \end{aligned}$$

(i) If $t \in [0, h]$ then

$$\begin{aligned} \|y_1(t-h)\|^2 \le& \|y(t-h)\|^2 = \varphi(t-h)^\top[G^{-1}]^\top[G^{-1}]\varphi(t-h) \\ &\le \frac{\lambda_{max}([G^{-1}]^\top[G^{-1}])}{\lambda_{min}(Q)}\varphi(t-h)^\top Q\varphi(t-h) \\ &\le \frac{\lambda_{max}([G^{-1}]^\top[G^{-1}])}{\lambda_{min}(Q)}c_1 = \alpha_3 c_1, \end{aligned}$$

(ii) if $t \in [h, T]$ then

$$\|y_1(t-h)\| \le \gamma_5 e^{0.5\beta T},$$

hence

$$\|y_1(t-h)\| \le \gamma_5 e^{0.5\beta T} + \sqrt{\alpha_3 c_1}, \qquad t \in [0, T].$$

Therefore, we derive from the conditions (8) and (21) that

$$||y_2(t)|| \le \alpha_4 + \alpha_5 ||y_2(t-h)||.$$

For $t \in [0, h]$, we have

$$\|y_2(t)\| \le \alpha_4 + \alpha_5 \sqrt{\alpha_3 c_1},$$

because of

$$\|y_2(t-h)\|^2 \le \|y(t-h)\|^2 = \varphi(t-h)^\top [G^{-1}]^\top [G^{-1}] \varphi(t-h) \le \alpha_3 c_1.$$

By induction, for $t \in [ih, (i+1)h] \cap [0, T],\ ih \le T,\ i = 0, 1, \ldots,$ we have

$$\|y_2(t)\| \le \alpha_4 \sum_{k=0}^{i} \alpha_5^k + \alpha_5^{i+1} \sqrt{\alpha_3 c_1},$$

and hence for all $t \in [0, T]$:

$$\|y_2(t)\| \le \max_{i=0,1,2,\ldots,[\frac{T}{h}]} (\alpha_4 \sum_{k=0}^{i} \alpha_5^k + \alpha_5^{i+1} \sqrt{\alpha_3 c_1}) \le \alpha_4 \gamma_6 + \gamma_7 = \sqrt{\alpha_6}. \tag{22}$$

Finally, combining Eqs. (20) and (22), for all $t \in [0, T]$ we obtain that

$$\begin{aligned} x(t)^\top Q x(t) &= y(t)^\top G^\top Q G y(t) \le \lambda_{max}(G^\top Q G) \|y(t)\|^2 \\ &= \lambda_{max}(G^\top Q G) \Big[\|y_1(t)\|^2 + \|y_2(t)\|^2 \Big] \le \gamma_8 \Big(\gamma_5^2 e^{\beta T} + \alpha_6 \Big). \end{aligned}$$

On the other hand, the LMI condition (9) is, by Proposition 2, equivalent to the following inequality

$$\alpha_7 e^{0.5\beta T} + \alpha_8 + \alpha_9 e^{\beta T} - c_2 < 0.$$

By simple computation we can verify that

$$\gamma_8 \Big(\gamma_5^2 e^{\beta T} + \alpha_6 \Big) = \alpha_7 e^{0.5\beta T} + \alpha_8 + \alpha_9 e^{\beta T},$$

and we finally get

$$x(t)^\top Q x(t) \le c_2, \quad \forall t \in [0, T].$$

This completes the proof of Theorem 1. □

Remark 1. The condition (8) in Theorem 1 involves the inverse of matrix A_{22}^l, but it can be seen from the proof of regularity and impulse-free of the system that this condition is derived from LMI conditions (5) and (6).

Remark 2. It is worth noting that strict LMI conditions are more desirable than non-strict ones such that the condition (5) cannot be solved by MATLABs LMI Toolbox. For tackling this, the matrix inequality (5) combined with Eqs. (6) and (7) can be reduced to strict LMIs.

Let P be a symmetric positive definite matrix, $S \in R^{n \times n-r}$ be any full-column rank matrix such that $E^\top S = 0$ and V be any matrix of appropriate dimension, denote $\bar{P} = (PE + SV)^\top$ then $\bar{P}E = E^\top \bar{P}^\top = E^\top PE \geq 0$. Thus, by changing P to $\bar{P} = (PE + SV)^\top$ in Eqs. (6) and (7), they are strict LMIs and can be solved by MATLAB LMI Toolbox.

Remark 3. We note that the condition (7) in Theorem 1 is a bilinear matrix inequality (BMI) with respect to $\xi_l, \;\; l = \overline{1, p}$ and $P, U, S_l, \;\; l = \overline{1, p}$. To find $\xi_l \geq 0$, the matrices $P, U, S_l, \;\; l = \overline{1, p}$ satisfying BMI (7) and LMIs (6), we can use the branch and bound methods proposed in [25] or the homotopy-based algorithm in [26]. Moreover, the condition (9) is an LMI w.r.t. $e^{0.5\beta T}$ and since β is not included in LMIs (5)–(7), we can easily determine β from LMI (9) w.r.t. $e^{0.5\beta T}$ using LMI Toolbox in Matlab [27].

Remark 4. In [17,19–21], finite-time stability for switched systems was investigated, but under arbitrary switching laws or switching laws specified by the average dwell-time. In this paper, we give constructive design of the switching laws via LMIs and constructing a specific Lyapunov function with less decision matrix variables, which reduces the computational complexity to some extent.

Example 1. Consider system (1), where

$$E = \begin{bmatrix} 2 & 0.5 \\ 0 & 0 \end{bmatrix}, \; M = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \; G = \begin{bmatrix} 0.5714 & -0.2857 \\ -0.2857 & 1.1429 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} -0.5 & -1 \\ 1.5 & -0.5 \end{bmatrix}, \; A_2 = \begin{bmatrix} 0.5 & 1 \\ 2.5 & 1.5 \end{bmatrix}, \; D_1 = \begin{bmatrix} 1.1 & 0.8 \\ 0.8 & 0.55 \end{bmatrix},$$

$$D_2 = \begin{bmatrix} 1.4 & 0.7 \\ 0.45 & 0.2 \end{bmatrix}, B_1 = \begin{bmatrix} 0.3 & 0.3 \\ 0.2 & 0.1 \end{bmatrix}, \; B_2 = \begin{bmatrix} 0.3 & 0.3 \\ 0.1 & 0.2 \end{bmatrix},$$

$$a_l = b_l = m_l = 0.01, \;\; l = 1, 2, \;\; h = 1, \;\; d = 0.1.$$

By using LMI Toolbox in Matlab, LMI (6) is feasible with

$$P = \begin{bmatrix} 3.4355 & -2.1132 \\ 0.8589 & 0.4717 \end{bmatrix}, \; U = \begin{bmatrix} 11.8143 & 3.3383 \\ 3.3383 & 2.9799 \end{bmatrix},$$

$$S_1 = \begin{bmatrix} 0.000 & -6.0127 \\ -6.0127 & 1.2294 \end{bmatrix}, \; S_2 = \begin{bmatrix} 0.000 & -5.8699 \\ -5.8699 & -4.8430 \end{bmatrix}.$$

In this case, it can be computed that

$$L_1 = \begin{bmatrix} -2.0723 & 4.7331 \\ 4.7331 & 1.2905 \end{bmatrix}, \; L_2 = \begin{bmatrix} -6.4058 & -6.1607 \\ -6.1607 & -2.6982 \end{bmatrix},$$

$$L_1 + L_2 = \begin{bmatrix} -8.4782 & -1.4276 \\ -1.4276 & -1.4076 \end{bmatrix} < 0.$$

Thus, the system of matrices $\{L_1, \; L_2\}$ is strictly complete. The sets $\overline{\Omega}_l$ are given as (see Fig. 1)

$$\overline{\Omega}_1 = \Big\{x = (x_1, x_2)^\top : \;\; (x_1 + 0.1325x_2)(x_1 - 4.7005x_2) \geq 0\Big\},$$

$$\overline{\Omega}_2 = \Big\{x = (x_1, x_2)^\top : \;\; (x_1 + 0.1325x_2)(x_1 - 4.7005x_2) < 0\Big\}.$$

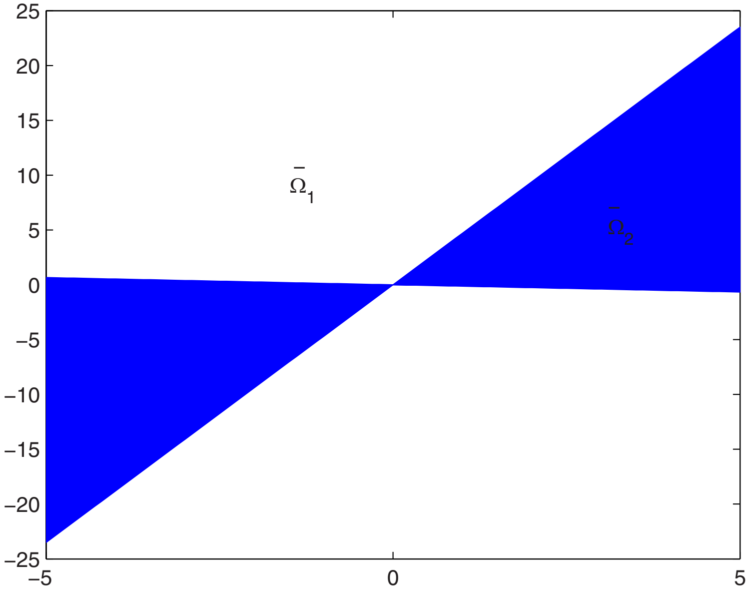

Fig. 1. The sets $\overline{\Omega_1}$ and $\overline{\Omega_2}$.

Moreover, we see that

$$PE = E^\top P^\top = \begin{bmatrix} 6.8711 & 1.7178 \\ 1.7178 & 0.4294 \end{bmatrix} \geq 0,$$

$$1 - a||G||. \max_{l=\overline{1,p}} \|[A_{22}^l]^{-1}\| = 1 - 0.01 \times 1.2612 \times 1 = 0.9874 > 0,$$

and the condition (9) holds with

$$\beta = 0.01,\ c_1 = 0.1,\ c_2 = 21.5,\ T = 5,\ Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Fig. 2 shows the state response of the switching function $\sigma(t)$. By Theorem 1, the system (1) with the switching rule $\sigma(x(t)) = l$ whenever $x(t) \in \overline{\Omega}_l$, is finite-time stable w.r.t. (0.1, 21.5, 5, Q). Fig. 3 shows the state response of the solution with the initial condition $\varphi(t) = (0.2857,\ 0.0572)^T \sin(\pi t), t \in [-1, 0]$.

We conclude this section with an application to robust finite-time stability of uncertain linear switched singular systems with delay considered in [12,28,29]. Consider the following uncertain linear switched time-delay system:

$$\begin{cases} E\dot{x}(t) = & [A_\sigma + \Delta A_\sigma]x(t) + [D_\sigma + \Delta D_\sigma]x(t-h) + [B_\sigma + \Delta B_\sigma]\omega(t), \\ x(\theta) = & \varphi(\theta),\ \ \theta \in [-h, 0], \end{cases} \tag{23}$$

where the time-varying uncertainties $\Delta A_l, \Delta D_l, \Delta B_l$ are given by

$$[\Delta A_l,\ \ \Delta D_l,\ \ \Delta B_l] = K_l H_l(t)[L_{A_l},\ \ L_{D_l},\ \ L_{B_l}], \quad l = \overline{1, p},$$

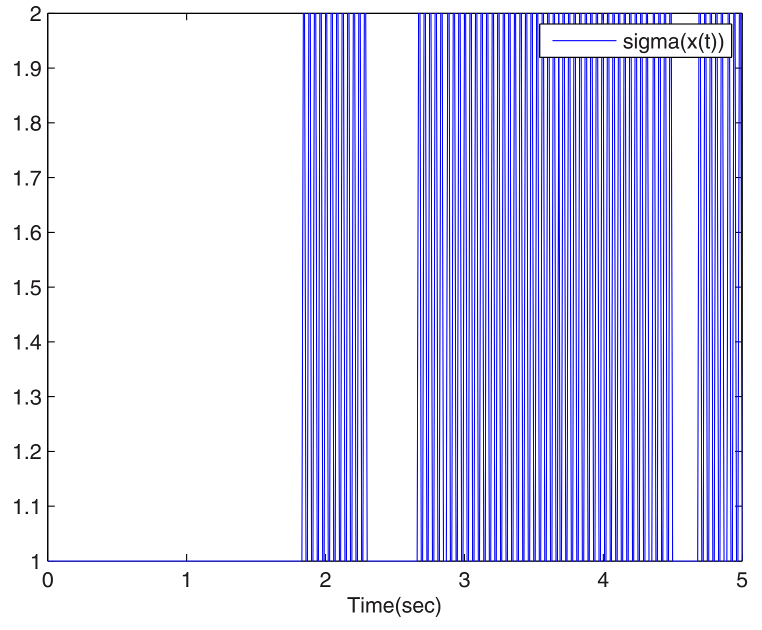

Fig. 2. The state response of $\sigma(t)$.

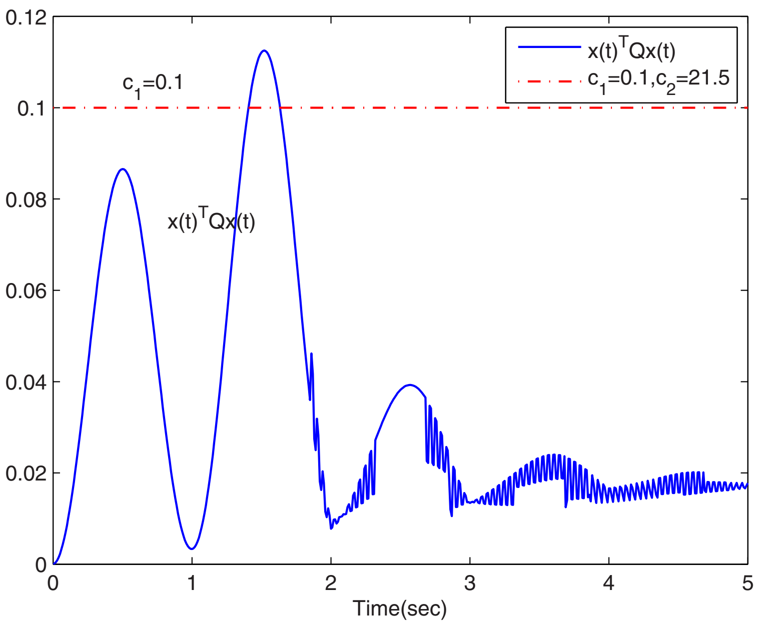

Fig. 3. The state response of $x(t)Qx(t)$.

$K_l,\ L_{A_l},\ L_{D_l},\ L_{B_l}$ are known real constant matrices of appropriate dimensions and $H_l(t)$ are unknown uncertain matrices satisfying

$$H_l(t)^\top H_l(t) \le I, \quad t \ge 0.$$

To apply Theorem 1, let us denote $f_l(t, x, x_h, \omega) = \Delta A_l x + \Delta D_l x_h + \Delta B_l \omega$. It is easy to verify that

$$||f_l(t, x, x_h, \omega)|| \le ||K_l||.||L_{A_l}||.||x|| + ||K_l||.||L_{D_l}||.||x_h|| + ||K_l||.||L_{B_l}||.||\omega||.$$

By the same notations used in Theorem 1 and applying Theorem 1 with

$$a_l = ||K_l||.||L_{A_l}||,\ \ b_l = ||K_l||.||L_{D_l}||,\ \ m_l = ||K_l||.||L_{B_l}||,$$

we have the following corollary, which gives sufficient conditions for designing the switching law for robust finite-time stability of the system (23).

Corollary 2. *For given positive numbers T, c_1, c_2, and a symmetric positive definite matrix $Q \in R^{n\times n}$, the system (22) is finite-time stable w.r.t. (c_1, c_2, T, Q) if there exist a symmetric positive definite matrix $U \in R^{n\times n}$, a non-singular matrix $P \in R^{n\times n}$, any matrices $S_l \in R^{n\times n}$, $l \in \overline{1,p}$, scalars $\xi_l \ge 0$, $l = \overline{1,p}$, $\sum_{l=1}^{p} \xi_l > 0$, and a number $\beta > 0$ such that the conditions (5)–(9) of Theorem 1 hold. The switching rule is chosen as $\sigma(x(t)) = l$ whenever $x(t) \in \overline{\Omega}_l$.*

Example 2. Consider system (22), where

$$E = \begin{bmatrix} 1 & 0 \\ 0.5 & 0 \end{bmatrix},\ M = \begin{bmatrix} 1 & 0 \\ -0.5 & 1 \end{bmatrix},\ G = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} -1 & 0 \\ -0.5 & -1 \end{bmatrix},\ A_2 = \begin{bmatrix} -1 & 1 \\ 1.5 & 1 \end{bmatrix},\ D_1 = \begin{bmatrix} 0.3 & 0.2 \\ 0.85 & 0.5 \end{bmatrix},$$

$$D_2 = \begin{bmatrix} 0.7 & 0.3 \\ 0.65 & 0.25 \end{bmatrix}, B_1 = \begin{bmatrix} 0.1 & 0.2 \\ 0.25 & 0.2 \end{bmatrix},\ B_2 = \begin{bmatrix} 0.2 & 0.1 \\ 0.2 & 0.25 \end{bmatrix},$$

$$K_1 = \begin{bmatrix} 0.09 & 0.03 \\ -0.02 & 0.09 \end{bmatrix},\ K_2 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix},$$

$$L_{A_1} = \begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{bmatrix},\ L_{D_1} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix},\ L_{B_1} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix},$$

$$L_{A_2} = \begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{bmatrix},\ L_{D_2} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix},\ L_{B_2} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix},$$

$$a_1 = 0.0985,\ \ a_2 = 0.1,\ \ b_l = m_l = 0.01,\ \ l = 1, 2,\ \ h = 1,\ \ d = 0.1.$$

By using LMI Toolbox in Matlab, LMI (6) is feasible with

$$P = \begin{bmatrix} 1.6129 & 2.0843 \\ -0.5000 & 1.0000 \end{bmatrix},\ U = \begin{bmatrix} 6.9190 & 2.3270 \\ 2.3270 & 2.2723 \end{bmatrix},$$

$$S_1 = \begin{bmatrix} 0.000 & -1.0727 \\ -1.0727 & 2.9808 \end{bmatrix},\ S_2 = \begin{bmatrix} 0.000 & -7.2408 \\ -7.2408 & -2.6133 \end{bmatrix}.$$

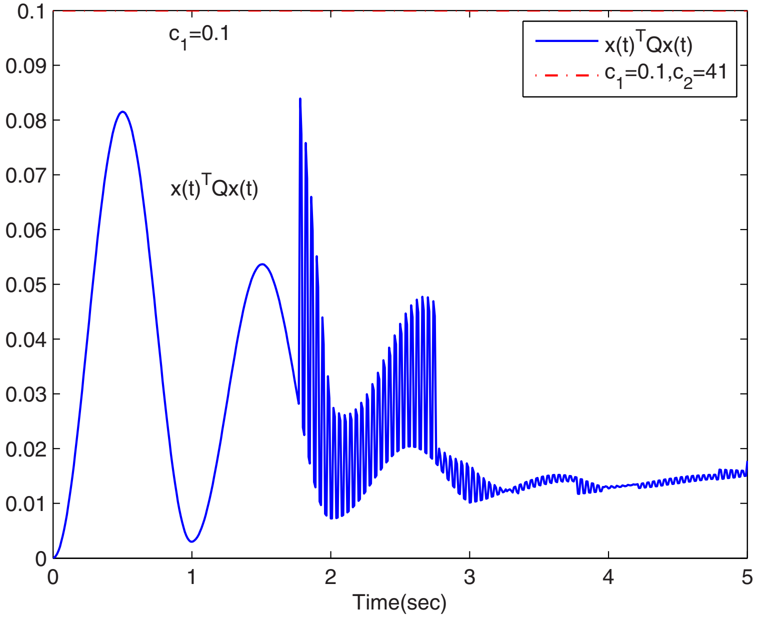

Fig. 4. The state response of $x(t)Qx(t)$.

In this case, we can find

$$L_1 = \begin{bmatrix} 4.4639 & 2.0212 \\ 2.0212 & -1.5086 \end{bmatrix}, \ L_2 = \begin{bmatrix} -5.8491 & -1.6645 \\ -1.6645 & 0.8589 \end{bmatrix},$$

$$L_1 + L_2 = \begin{bmatrix} -1.3852 & 0.3567 \\ 0.3567 & -0.6496 \end{bmatrix} < 0,$$

such that the system of matrices $\{L_1, L_2\}$ is strictly complete. The sets $\overline{\Omega}_l$ are given as

$$\overline{\Omega}_1 = \left\{x = (x_1, x_2)^\top : \ (x_1 - 0.2691x_2)(x_1 + 1.2558x_2) \le 0\right\},$$

$$\overline{\Omega}_2 = \left\{x = (x_1, x_2)^\top : \ (x_1 - 0.2691x_2)(x_1 + 1.2558x_2) > 0\right\}.$$

Moreover, we see that

$$PE = E^\top P^\top = \begin{bmatrix} 2.6551 & 0 \\ 0 & 0 \end{bmatrix} \ge 0,$$

$$1 - a||G||. \max_{l=\overline{1,p}} \|[A_{22}^l]^{-1}\| = 1 - 0.1 \times 1.6180 \times 1 = 0.8382 > 0,$$

and the condition (9) holds with

$$\beta = 0.001, \ c_1 = 0.1, \ c_2 = 41, \ T = 5, \ Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

By Corollary 2, the system (23) with the switching rule $\sigma(x(t)) = l$ whenever $x(t) \in \overline{\Omega}_l$, is finite-time stable w.r.t. (0.1, 41, 5, Q). Fig. 4 shows the state response of the solution with the initial function $\varphi(t) = (0.2,\ 0.2)^T sin(\pi t), t \in [-1, 0]$.

4. Conclusions

In this paper, we have studied the finite-time stability of singular nonlinear switched systems with delays and disturbances. By using state-space singular value decomposition approach combining with Lyapunov-like function method, the proposed finite-time stability criteria have been established in the terms of LMIs. The proposed approach allowed us to apply the obtained results to finite-time stability for uncertain singular linear switched time-delay systems. The extension of the proposed method to the time-varying delay systems is to be considered in the future.

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References

[1] D. Liberzon, Switching in Systems and Control, Brikhauser, Boston, 2003.
[2] Z.D. Sun, Switched Linear Systems: Control and Design, Springer, New York, 2005.
[3] J. Zhang, X. Zhao, J. Huang, Absolute exponential stability of switched nonlinear time-delay systems, J. Frankl. Inst. 353 (2016) 1249–1267.
[4] C. Liu, W. Liu, X. Liu, C. Li, Q. Han, Stability of switched neural networks with time delay, Nonlinear Dyn. 79 (2015) 2145–2154.
[5] V.N. Phat, Switched controller design for stabilization of nonlinear hybrid systems with time-varying delays in state and control, J. Frankl. Inst. 347 (2010) 195–207.
[6] S.L.V. Campbell, Singular Systems of Differential Equations-Research Notes in Mathematics, 1982.
[7] F.L. Lewis, A survey of linear singular systems, Circuits Syst. Signal Process. 5 (1986) 3–36.
[8] L. Dai, Singular Control Systems, Springer, New York, 1989.
[9] S. Xu, J. Lam, Robust Control and Filtering of Singular Systems, Springer, Berlin, 2006.
[10] M. Chadli, H.R. Karimi, P. Shi, On stability and stabilization of singular uncertain Takagi–Sugeno fuzzy systems, J. Frankl. Inst. 351 (2014) 1453–1463.
[11] L. Zhou, D.W.C. Ho, G. Zhai, Stability analysis of switched linear singular systems, Automatica 49 (2013) 1481–1487.
[12] L.J. Xing, F.S. Min, Robust exponential admissibility of uncertain switched singular time-delay systems, Acta Autom. Sin. 36 (2010) 1773–1779.
[13] R. Krishnasamy, P. Balasubramaniam, A descriptor system approach to the delay-dependent exponential stability analysis for switched neutral systems with nonlinear perturbations, Nonlinear Anal. Hybrid Syst. 15 (2015) 23–36.
[14] I. Zamani, M. Shafiee, A. Ibeas, Exponential stability of hybrid switched nonlinear singular systems with time-varying delay, J. Frankl. Inst. 350 (2013) 171–193.
[15] P. Dorato, Short time stability in linear time-varying systems, Proc. IRE Int. Conv. Record 4 (1961) 83–87.
[16] F. Amato, R. Ambrosino, M. Ariola, C. Cosentino, Finite-Time Stability and Control, Lecture Notes in Control and Information Sciences, 453, Springer, New York, 2014.
[17] J. Cheng, H. Zhu, S. Zhong, F. Zheng, Y. Zeng, Finite-time filtering for switched linear systems with a mode-dependent average dwell time, Nonlinear Anal. Hybrid Syst. 15 (2015) 145–156.
[18] Z. Zhang, Z. Zhang, H. Zhang, Finite-time stability analysis and stabilization for uncertain continuous-time system with time-varying delay, J. Frankl. Inst. 352 (2015) 1296–1317.
[19] X. Li, X. Lin, S. Li, Y. Zou, Finite-time stability of switched nonlinear systems with finite-time unstable subsystems, J. Frankl. Inst. 352 (2015) 1192–1214.

[20] Y. Wang, Y. Zou, Y. Liu, X. Shi, Z. Zuo, Average dwell-time approach to finite-time stabilization of switched singular linear systems, J. Frankl. Inst. 352 (2015) 2920–2933.
[21] Y. Xia, E.-K. Boukas, P. Shi, J. Zhang, Stability and stabilization of continuous-time singular hybrid systems, Automatica 45 (2009) 1504–1509.
[22] G. Zhang, S. Wang, Y. Wang, Y. Lin, Finite-time stability for singular linear time-delay systems with time–varying exogenous disturbance, in: Proceedings of the Twenty-fifth Chinese Control and Decision Conference, 2013, pp. 2583–2586.
[23] F. Uhlig, A recuring theorem about pairs of quadratic forms and extensions, Linear Algebra Appl. 25 (1979) 219–237.
[24] S. Boyd, E. Ghaoui, E. Feron, V. Balakrishnan, Linear Matrix Inequalities in System and Control Theory, SIAM, Philadelphia, 1994.
[25] K.C. Goh, M.G. Safonov, G.P. Papavassilopoulos, A global optimization approach for the BMI problem, in: Proceedings of the 1994 IEEE Conference on Decision and Control, Orlando, FL, USA, 1994, pp. 2009–2014.
[26] G. Zhai, M. Ikeda, Y. Fujisaki, Decentralized h_∞ controller design: a matrix inequality approach using a homotopy method, Automatica 37 (2001) 565–572.
[27] P. Gahinet, A. Nemirovskii, A.J. Laub, M. Chilali, LMI Control Toolbox For use with Matlab, The MathWorks, Inc., Massachusetts, 1985.
[28] Y. Zhao, J. Yang, G. Wen, h_∞ control for uncertain switched singular systems with time delay, Nonlinear Dyn. 74 (2013) 649–665.
[29] Y. Ding, F. Weng, W. Mao, Improved stability and stabilization criteria of singular system with time-delay and parameter uncertainties, J. Inf. Comput. Sci. 10 (2013) 6067–6076.