IET Control Theory & Applications

Research Article

Stability analysis of fractional differential time-delay equations

ISSN 1751-8644
Received on 23rd August 2016
Revised 15th November 2016
Accepted on 27th January 2017
E-First on 1st March 2017
doi: 10.1049/iet-cta.2016.1107
www.ietdl.org

Nguyen T. Thanh[1], Hieu Trinh[2] ✉, Vu N. Phat[3]

[1]*Department of Mathematics, University of Mining and Geology, Hanoi, Vietnam*
[2]*School of Engineering, Deakin University, Geelong 3217, VIC, Australia*
[3]*Institute of Mathematics, VAST, 18 Hoang Quoc Viet Road, Hanoi, Vietnam*
✉ *E-mail: hieu.trinh@deakin.edu.au*

Abstract: This study provides a novel analytical approach to studying the solutions and stability of fractional differential delay equations without using Lyapunov function method. By applying the properties of Caputo fractional derivatives, the Laplace transform and the Mittag–Leffler function, the authors first provide an explicit formula and solution bounds for the solutions of linear fractional differential delay equations. Then, they prove new sufficient conditions for exponential boundedness, asymptotic stability and finite-time stability of such equations. The results are illustrated by numerical examples.

1 Introduction

During the past decades, stability theory of fractional differential equations (FDEs) has attracted much attention from mathematicians and engineers in various sciences such as electrical engineering, bioengineering, control theory, acoustics, optics, chemical physics [1–6] etc. In fact, many real-world physical systems are well characterised by FDEs, i.e. equations involving non-integer-order derivatives. These new fractional order models are more accurate than integer-order models and provide an excellent instrument for the description of memory and hereditary processes. Since the fractional derivative has the non-local property and weakly singular kernels, the analysis of stability of FDEs is more complicated than that of integer-order differential systems. Also, we cannot directly use algebraic tools for fractional order systems since for such a system we do not have a characteristic polynomial, but rather a pseudo-polynomial with a rational power-multivalued function. On the other hand, time delay has an important effect on the stability and performance of dynamic systems. The existence of a time delay may cause undesirable system transient response, or generally, even an instability. The unification of differential delay equations and functional differential equations (DE) is provided by fractional differential delay equations, involving both the delay and non-integer derivative terms and disposing great complexity. The corresponding stability polynomials have infinitely many isolated zeros and analysis of their location is often a complicated matter. On the other hand, stability analysis of the linear fractional differential delay equations (LFDDEs) $D^{\alpha}x(t) = Ax(t) + Bx(t-h)$ is more complicated because asymptotic stability of such systems is equivalent to asymptotic stability of the corresponding infinite-dimensional systems of natural order with delays, and due to the presence of the exponential function e^{-sh}, this equation has an infinite number of roots, which makes the analytical stability analysis of a time-delay system extremely difficult.

Over the past years, the solutions and stability analysis of LFDDEs have attracted attention of many mathematicians and engineers [7–15]. There are two main approaches in studying the solutions and stability of FDDEs. One of them is the Hale analytical approach based on Lyapunov function method for functional DEs [16]. The other is based on the fractional derivative calculus (Caputo fractional derivatives, the Laplace transform and the Mittag–Lefller function [17]). Ye *et al.* [7] and Lazarevi and Spasi [8] studied asymptotic stability of LFDDEs by using a Gronwall inequality approach. In [9], Deng *et al.* studied asymptotic stability of LFDDEs by using the final-value theorem of the Laplace transform. Johnson [10] considered stability and instability of Korteweg and de Vries equation by using non-local Floquet-like theory and spectral perturbation theory. On the basis of Lambert function approach, an analytical stability bound was derived in [11] for a class of LFDDEs. In [12], robust stability of LFDDEs by means of fixed point theory was considered. A survey on the stability of LFDDEs was presented in [13]. On the basis of the algebraic approach and numerical methods, Cermak *et al.* [14] and Kaslik and Sivasundaram [15] proposed some criteria for asymptotic stability of singular LFDDEs. For the fractional systems without delays, using the analytical approach Li *et al.* [18] constructed Lyapunov functional to study asymptotic stability of linear FDEs without delays. Then, the authors of [19–24] extended the method of [18] to LFDDEs and proposed Lyapunov–Krasovskii and Razumikhin stability theorems. Designing a positive Lyapunov function is a key problem of these methods. However, it is usually very difficult to construct a positive function according to the provided fractional systems, especially for fractional time-delay systems. Furthermore, in many cases, the use of the trace of matrices inside the Lyapunov functions can be useful in proving the stability of ordinary DEs (ODEs), but there is no well established result for LFDDEs, even there does not exist such Lyapunov functions. Other disadvantage of the Lyapunov function method is the computational difficulty in solving linear matrix inequality (LMI) conditions. To the best of the authors' knowledge, up to now, how to analyse the stability LFDDEs is still an open and challenging problem. Moreover, as noted by many authors (see, e.g. [6, 9, 11]), the existing stability conditions for LFDDEs do not provide effective algebraic criteria or algorithms for testing the stability of LFDDEs and they are difficult to use in practise. In addition, a strong motivation for investigating the stability analysis of LFDDEs is an explicit expression of their solutions. Some analytical properties of the solutions of LFDDEs were proposed using the Laplace transform via eigenvalues of the system matrix and their location in a specific area of the complex plane can be found in [25–28]. In [25, 28], the authors proposed an explicit formula of the solutions to a special linear FDE ($A = 0$) and based on this formula some sufficient conditions for the asymptotic stability, finite-time stability (FTS) were derived under some restricted assumptions. However, to the best of our knowledge, no explicit formulas of solutions of LFDDEs exist in the literature.

Motivated by the above discussion, this paper is devoted to study the solutions and stability analysis of LFDDEs by a newly proposed method which is a combination of the analytical Hale's approach and the fractional derivative calculus. The main contribution of this paper has two points. First, based on the proposed approach we provide an explicit form of the solutions of linear LFDDEs. Using this solution formula, we estimate more precisely exponential bounds of the solution via the roots of the characteristic equations. Second, we derive new sufficient conditions for the asymptotic stability and FTS of such equations neither using the Lyapunov method nor LMI technique. This paper is organised as follows. In Section 2, we present some basic definitions and some well-known technical propositions, which are needed for the proof of the main results. Main results on explicit formula of solutions, exponential bounds of the solutions and applications of the results to asymptotic stability, FTS of LFDDEs are presented in Section 3 with numerical examples. Finally, Section 4 concludes this paper.

2 Problem formulation and preliminaries

The following notations will be used throughout this paper. $\mathbb{N}^+$ denotes the set of all positive integer numbers; $\mathbb{C}$ denotes the complex space; R^+ denotes the set of all real positive numbers; R^n denotes the n- dimensional space; $R^{n\times r}$ denotes the space of all $(n\times r)$- matrices; the notation $i=\overline{1,p}$ means $i=1,2,\dots,p$; $\lambda(A)$ denotes the set of all eigenvalues of A; $\lambda_{\max}(A)=\max\{\mathrm{Re}(\lambda)\colon \lambda\in\lambda(A)\}$; $\lambda_{\min}(A)=\min\{\mathrm{Re}(\lambda)\colon \lambda\in\lambda(A)\}$; $C([-h,0],R^n)$ denotes the set of all R^n-valued continuously functions on $[-h,0]$ with the norm $\|\varphi\| = \sup_{\theta\in[-h,0]}\|\varphi(\theta)\|$; $0(s)$ denotes the infinitesimal function of higher order with respect to (w.r.t.) s; the spectrum set of A,B w.r.t. $\alpha>0$ is denoted by $\sigma_{A,B}^{\alpha,h}:=\{s\in\mathbb{C}\colon \det(s^\alpha I-A-e^{-sh}B)=0\}$. D^α denotes the Caputo fractional derivative of order α for the function $f^{(n)}(t)\in L^1[0,T]$ defined as

$$D^\alpha f(t)=\frac{\mathrm{d}^\alpha f(t)}{\mathrm{d}t^\alpha}=\frac{1}{\Gamma(n-\alpha)}\int_0^t\frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}}\,\mathrm{d}\tau$$

where $0\le n-1\le\alpha<n$, $n\in\mathbb{N}^+$ and $\Gamma(z)$ is the Gamma function defined by $\Gamma(z)=\int_0^\infty e^{-t}t^{z-1}\,\mathrm{d}t,\ z\in\mathbb{C}$.

Proposition 1 [1, 3]: The Gamma function satisfies condition

$$\frac{t^p}{\Gamma(p+1)}=\frac{1}{2\pi i}\lim_{T\to\infty}\int_{d-iT}^{d+iT}e^{st}\frac{1}{s^{p+1}}\,\mathrm{d}s,\quad p>-1$$

where the integration is done along the vertical line $\mathrm{Re}(s)=d>0$.

The Mittag–Leffler function is defined by $E_\alpha(z)=\sum_0^{+\infty}\frac{z^k}{\Gamma(\alpha k+1)},\ \alpha>0$.

Proposition 2 [17]: For $0<\alpha<2$, $|\arg(z)|<0.5\alpha\pi$, the Mittag–Leffler function $E_\alpha(z)$ satisfies the condition

$$E_\alpha(z)=\frac{1}{\alpha}\exp\{z^{1/\alpha}\}+0(z^{-1}),\quad \text{for } z\to\infty$$

The Laplace transform of Caputo fractional derivative $D^\alpha f(t)$, $0<\alpha<1$, is given by

$$\mathscr{L}[D^\alpha f(t)](s)=s^\alpha\mathscr{L}[f(t)]-s^{\alpha-1}f(0)$$

where $\mathscr{L}[f(t)](s)$ is the Laplace transform of function $f(t)$ defined as

$$\mathscr{L}[f(t)](s)=\int_0^{+\infty}e^{-st}f(t)\,\mathrm{d}t,\quad s\in\mathbb{C}$$

Proposition 3 [16]: If the function $x(t)$ is exponentially bounded, i.e. $\exists a,b\in R\colon ||x(t)||\le ae^{bt}$, $\forall t\ge0$, then the Laplace transformation of $x(t)$ is well-defined and analytic on $\{s\colon\mathrm{Re}(s)>b\}$.

Consider the following LFDDE

$$\begin{cases}D^\alpha x(t)=Ax(t)+Bx(t-h),\ t\ge0,\\ x(\theta)=\varphi(\theta),\ \theta\in[-h,0]\end{cases}\tag{1}$$

where the state $x(t)\in R^n$; $h>0$; $0<\alpha<1$; $A\in R^{n\times n}, B\in R^{n\times n}$ are given constant matrices; $\varphi(t)\in C([-h,0],R^n)$ is the given initial function.

Definition 1: A continuous function $x(t)\colon[0,\infty)\to R^n$ satisfying the equation

$$\begin{cases}x(t)=x(0)+\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t(t-\tau)^{\alpha-1}[Ax(\tau)+Bx(\tau-h)]\,\mathrm{d}\tau,\ t\ge0,\\ x(\theta)=\varphi(\theta),\ \theta\in[-h,0]\end{cases}\tag{2}$$

is called a mild solution of (1).

Proposition 4 [29, 30]: Assume that the initial function $\varphi(t)\in C([-h,0],R^n)$, then the LFDDE (1) has a unique mild solution.

Definition 2 (Lyapunov stability): The system (1) is said to be (i) stable if for every $\epsilon>0$ there exists $\delta>0$ such that $\|x(t,\varphi)\|<\epsilon$ provided $\|\varphi\|<\delta$; (ii) asymptotically stable if it is stable and $\lim_{t\to\infty}\|x(t,\varphi)\|=0$.

Definition 3 (FTS): Given positive numbers c_1,c_2,T, where $c_1<c_2$. The system (1) is said to be finite-time stable w.r.t. (c_1,c_2,T), if $\|\varphi\|\le c_1$ implies $\|x(t,\varphi)\|\le c_2$, for all $t\in[0,T]$.

Remark 1: Different from Lyapunov stability, FTS deals with systems whose operation is limited to a fixed finite interval of time. From practical considerations, FTS seems to be more appropriate for systems whose variables must lie within specific bounds. The problem of FTS of fractional-order systems with time delay was considered in [8, 28, 31–33].

Proposition 5: Assume that matrices $\{C_m\}$, $C\in\mathbb{C}^{n\times n}$ are invertible and $\lim_{m\to\infty}||C_m-C||=0$. We have

$$\lim_{m\to\infty}||C_m^{-1}||=||C^{-1}||,$$
$$\lim_{m\to\infty}||C_m||=||C||,$$
$$\lim_{m\to\infty}||C_m^{-1}-C^{-1}||=0$$

Proof: Using the identity

$$C_m^{-1}-C^{-1}=C^{-1}(C-C_m)C_m^{-1}\tag{3}$$

gives

$$\big|||C_m^{-1}||-||C^{-1}||\big|\le||C_m^{-1}-C^{-1}||\le||C-C_m||||C_m^{-1}||||C^{-1}||$$

Since C_m,C are invertible, we have $||C_m^{-1}||||C^{-1}||\ne0$, which gives

$$\big|1/||C_m^{-1}||-1/||C^{-1}||\big|\le||C-C_m||\to0\quad\text{as } m\to\infty$$

and hence, $\lim_{m\to\infty}||C_m^{-1}||=||C^{-1}||$. Therefore, there exists a number $M>0$ such that

$$\max\left\{\sup_m ||C_m^{-1}||,\ ||C^{-1}||\right\} < M < +\infty$$

Then

$$\begin{aligned}||C_m^{-1} - C^{-1}|| &= ||C^{-1}||\,||(C - C_m)||\,||C_m^{-1}|| \\ &\le M^2||(C - C_m)|| \to 0\end{aligned}$$

as $m \to \infty$. Therefore, $\lim_{m\to\infty} ||C_m^{-1} - C^{-1}|| = 0$. Finally, the inequality $\big|\,||C_m|| - ||C||\,\big| \le ||C_m - C||$ leads to $\lim_{m\to\infty} ||C_m|| = ||C||$. This completes the proof . □

Proposition 6: For given $\alpha \in (0, 1)$, the following conditions hold:

i. For given $\sigma \in R$, there exists $T_0 > 0$ such that $\det(s^\alpha - A - e^{-sh}B) \ne 0$, $\forall s = \sigma + iT$, $|T| \ge T_0$ and

$$(a)\ ||(s^\alpha I - A - e^{-sh}B)^{-1}|| \le \frac{1}{|s|^\alpha I - ||A|| - e^{-\sigma h}||B||}.$$

$$(b)\ (s^\alpha I - A - e^{-sh}B)^{-1} = \frac{1}{s^\alpha} + 0(s)\frac{1}{s^{2\alpha}},$$

$$||0(s)|| \le \frac{||A|| + e^{-\sigma h}||B||}{|T|^\alpha - ||A|| - e^{-\sigma h}||B||}$$

ii. If $\det(s^\alpha I - A - e^{-sh}B) \ne 0$ at $s_0 \notin (-\infty, 0]$, then $(s^\alpha I - A - e^{-sh}B)^{-1}$ is analytic at s_0.

Proof:

i. We choose $T_0 > 0$ such that $T_0^\alpha > ||A|| + e^{-\sigma h}||B||$. Hence, for $|T| \ge T_0$, we have

$$\begin{aligned}|s|^\alpha &= (\sigma^2 + T^2)^{\alpha/2} \ge T_0^\alpha > ||A|| + e^{-\sigma h}||B|| \\ &= ||A + e^{-sh}B||\end{aligned}$$

Note that if $||C|| < 1$, then $(I - C)^{-1} = \sum_{k=0}^\infty C^n$, which shows the existence of $(s^\alpha I - A - e^{-sh}B)^{-1}$ defined as

$$(s^\alpha I - A - e^{-sh}B)^{-1} = \sum_{k=0}^\infty \frac{(A + e^{-sh}B)^k}{s^{\alpha(k+1)}}$$

Then, we have

$$\begin{aligned}||(s^\alpha I - A - e^{-sh}B)^{-1}|| &\le \sum_{k=0}^\infty \frac{(||A|| + e^{-ch}||B||)^k}{|s|^{\alpha(k+1)}} \\ &= \frac{1}{|s|^\alpha - ||A|| - e^{-ch}||B||}\end{aligned}$$

and

$$||(s^\alpha I - A - e^{-sh}B)^{-1}|| \le \frac{1}{|T|^\alpha - ||A|| - e^{-ch}||B||}$$

The condition (b) is similarly proved.

ii. This condition is easily derived by using Proposition 5 and the identity (3).

This completes the proof .

□

Proposition 7 (Generalised Gronwall inequality [7]): Suppose that $\alpha > 0$, $a(t)$ is a non-negative function locally integrable on $[0, T)$, $g(t)$ is a non-negative, non-decreasing continuous function defined on $[0, T)$, $u(t)$ is a non-negative locally integrable function on $[0, T)$ satisfying the inequality

$$u(t) \le a(t) + g(t)\int_0^t (t - \tau)^{\alpha-1}u(\tau)\,d\tau, \quad 0 \le t < T$$

then

$$u(t) \le a(t) + \int_0^t \Big[\sum_{n=1}^\infty \frac{(g(t)\Gamma(\alpha))^n}{\Gamma(n\alpha)}(t - \tau)^{n\alpha-1}a(\tau)\Big]d\tau,$$
$$0 \le t < T$$

Moreover, if $a(t)$ is a non-decreasing function on $[0, T)$, then

$$u(t) \le a(t)E_\alpha(g(t)\Gamma(\alpha)t^\alpha), \quad t \ge 0$$

3 Main results

In this section, we first prove the exponential boundedness of solutions and then we give an explicit formula of the solution, exponential estimates of the solution of system (1). On the basis of this estimation, we will derive new conditions for the Lyapunov stability and FTS of the system.

Theorem 1: The mild solution $x(t)$ of (1) is exponentially bounded.

Proof: In view of the expression of the solution (2), we have

$$\begin{aligned}||x(t)|| &\le ||x(0)|| \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t (t - \tau)^{\alpha-1}[||A||\,||x(\tau)|| + ||B||\,||x(\tau - h)||]\,d\tau\end{aligned}$$

which implies

$$\|x(t)\| \le ||\varphi|| + \frac{1}{\Gamma(\alpha)}\int_0^t (t - \tau)^{\alpha-1}[||A|| + ||B||]||x_\tau||\,d\tau$$

where $\sup_{s\in[-h,t]} \|x(s)\| := \|x_t\|$. After some calculations, we have

$$\|x_t\| \le ||\varphi|| + \frac{||A|| + ||B||}{\Gamma(\alpha)}\int_0^t (t - \tau)^{\alpha-1}||x_\tau||\,d\tau, \quad t \ge 0$$

Using the generalised Gronwall inequality, Proposition 7, gives

$$||x_t|| \le ||\varphi||E_\alpha([(||A|| + ||B||]t^\alpha), \quad t \ge 0 \qquad (4)$$

Note that if $x \in R^+$, then arg $(x) = 0$, we can apply Proposition 2 for the case $0 < \alpha < 1$, to get

$$E_\alpha(x) = \frac{1}{\alpha}\exp\{x^{1/\alpha}\} + 0(x^{-1}) \quad \text{for } x \to \infty$$

Thus, there is a number $\eta > 0$ such that

$$E_\alpha(x) \le \frac{\eta}{\alpha}\exp\{x^{1/\alpha}\}, \quad x \ge 0 \qquad (5)$$

Therefore, we have

$$||x(t)|| \le \|x_t\| \le \frac{\eta}{\alpha}||\varphi||\exp\{(||A|| + ||B||)^{1/\alpha}t\}$$

which gives

$$||x(t)|| \le ae^{bt}, \quad t \ge 0$$

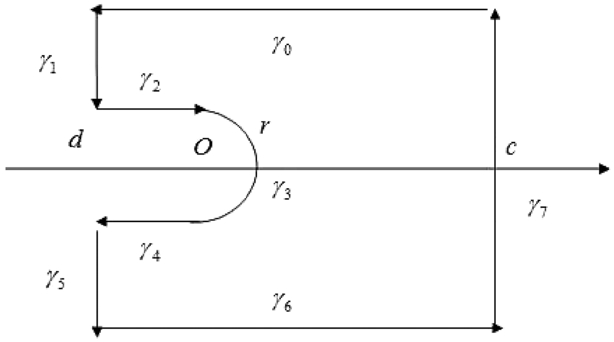

Fig. 1 *Boundary $\mathscr{D}$*

where $a = (\eta/\alpha)||\varphi||, b = (||A|| + ||B||)^{1/\alpha}$. This completes the proof of Theorem 1. □

Our next theorem gives an explicit formula of the solution $x(t)$ of system (1), the proof of which is based on the idea of [[16], Theorem 5.2]. Let us denote

$$R^{\alpha}_{A,B} = \sup\{\mathrm{Re}(s): s \in \sigma^{\alpha,h}_{A,B}\},$$
$$\Delta_{s,h}(A,B) = (s^{\alpha}I - A - e^{-sh}B)^{-1}$$
$$g_2(s) = \Delta_{s,h}(A,B)\left[g_1(s) + x(0)s^{\alpha-1}\right],$$
$$g_1(s) = \int_0^h e^{-s\tau}B\varphi(\tau - h)\,\mathrm{d}\tau$$
$$K(d,t) = \int_{-\infty}^{\infty} e^{i\sigma t}g_2(d + i\sigma)i\,\mathrm{d}\sigma$$

Theorem 2: Assume that $R^{\alpha}_{A,B} < d_0$.

i. If $d_0 \le 0$, then for all $d \in [d_0, 0]$ the mild solution of (1) is given by

$$x(t) = \frac{1}{2\pi i}\int_d^0 [e^{\sigma e^{-i\pi}t - i\pi}g_2(\sigma e^{-i\pi}) - e^{\sigma e^{i\pi}t + i\pi}g_2(\sigma e^{i\pi})]\,\mathrm{d} + \frac{1}{2\pi i}e^{\mathrm{d}t}K(d,t) \tag{6}$$

ii. If $d_0 > 0$, then for all $d \ge d_0$ the mild solution of (1) is given by

$$x(t) = \frac{1}{2\pi i}e^{\mathrm{d}t}K(d,t), \quad t \ge 0 \tag{7}$$

Proof: Since the solution $x(t)$ is, by Theorem 1, exponentially bounded, the Laplace transformation $X(s)$ of the solution $x(t)$ exists and is analytic on $\{s: \mathrm{Re}(s) > b\}$ by Proposition 3. The Laplace transformation applied to each term in equation (2) gives

$$(s^{\alpha}I - A - e^{-sh}B)X(s) = \int_0^h e^{-st}B\varphi(t-h)\,\mathrm{d}t + x(0)s^{\alpha-1}$$

such that for some fixed $c >$ max $\{0, b, d\}$, $\mathrm{Re}(s) > c$, the mild solution of (1) is given by

$$x(t) = \lim_{T \to +\infty}\frac{1}{2\pi i}\int_{c-iT}^{c+iT} e^{st}g_2(s)\,\mathrm{d}s = \lim_{T \to +\infty}\frac{1}{2\pi i}\int_{\gamma_7} e^{st}g_2(s)\,\mathrm{d}s \tag{8}$$

where $\gamma_7 = \{s = c + i\sigma: -T \le \sigma \le T\}$.

i. Let $d_0 \le 0$ and we consider the integration of the function $e^{st}g_2(s)$ around the closed boundary of the domain $\mathscr{D} = \bigcup_{i=0}^{7}\gamma_i$ (see Fig. 1) in the complex plane in the direction indicated

$$\gamma_0 = \{s = \sigma + iT: d \le \sigma \le c\}, \quad \gamma_6 = \{s = \sigma - iT: d \le \sigma \le c\},$$
$$\gamma_1 = \{s = d + i\sigma: r \le \sigma \le T\}, \quad \gamma_5 = \{s = d - i\sigma: r \le \sigma \le T\},$$
$$\gamma_2 = \{s = \sigma + ir: d \le \sigma \le 0\}, \quad \gamma_4 = \{s = \sigma - ir: d \le \sigma \le 0\},$$
$$\gamma_3 = \left\{s = re^{iv}: -\frac{\pi}{2} \le v \le \frac{\pi}{2}\right\}$$

Since $e^{st}g_2(s)$ is analytic in the domain $\mathscr{D}$ and has no zeros in this domain, it follows that the integral around its boundary is zero

$$\oint_{\mathscr{D}} e^{st}g_2(s)\,\mathrm{d}s = 0 \tag{9}$$

Step 1: *Estimation of $\int e^{st}g_2(s)\,\mathrm{d}s$ over γ_0 and γ_6 as $T \to \infty$:* Using Proposition 2, for given c there is a number $T_0 > 0$ such that for $s = c + iT, |T| > T_0$ the condition (i) in Proposition 6 holds, and hence we have for all $\sigma \in [d, c]$

$$\left\| \int_{\gamma_0} e^{st}g_2(s)\,\mathrm{d}s \right\| \le \int_{\gamma_0} |e^{(\sigma+iT)t}|\,||\Delta_{s,h}(A,B)||\left[||g_1(s)|| + ||x(0)||\,|s|^{\alpha-1}\right]|\mathrm{d}s|$$

$$= \frac{he^{-dh}||B||\,||\varphi|| + ||x(0)||T^{\alpha-1}}{T^{\alpha} - ||A|| - e^{-dh}||B||}|c-d|e^{ct}$$

because of $||g_1(s)|| \le \int_0^h |e^{-(\sigma+iT)t}|\,||B||\,||\varphi(t-h)||dt \le he^{-dh}||B||\,||\varphi||$.

Therefore, we have

$$\lim_{T\to\infty}\left\| \int_{\gamma_0} e^{st}g_2(s)\,\mathrm{d}s \right\| = 0 \tag{10}$$

Similarly, we can get

$$\lim_{T\to\infty}\left\| \int_{\gamma_6} e^{st}g_2(s)\,\mathrm{d}s \right\| = 0 \tag{11}$$

Step 2: *Estimation of $\int e^{st}g_2(s)\,\mathrm{d}s$ over γ_3 as $r \downarrow 0$:* We first note that

$$\lim_{r\downarrow 0,\ -(\pi/2) \le v \le (\pi/2)} s^{\alpha} = 0, \text{ where } s = re^{iv},\ s^{\alpha} = r^{\alpha}e^{iv\alpha}$$

From Proposition 2 it follows that

$$\lim_{r\downarrow 0,\ -(\pi/2) \le v \le (\pi/2)} (s^{\alpha}I - A - e^{-sh}B)^{-1} = (-A - B)^{-1}$$

On the other hand, we see that

$$\left\| \int_{\gamma_3} e^{st}g_2(s)\,\mathrm{d}s \right\| \le \int_{-\pi/2}^{\pi/2} e^{rt}||\Delta_{s,h}(A,B)||\left[h||B||\,.\,||\varphi||\,r + ||x(0)||r^{\alpha}\right]\mathrm{d}v$$

which gives $\lim_{r\downarrow 0}\left\|\int_{\gamma_3} e^{st}g_2(s)\,\mathrm{d}s\right\| = 0$, because of $||g_1(s)|| \le h||B||\,||\varphi||$.

Step 3: *Estimation of $\int e^{st}g_2(s)\,\mathrm{d}s$ over γ_2 and γ_4 as $r \downarrow 0$:* Since

$$\left\| \int_{\gamma_2} e^{st} g_2(s)\,\mathrm{d}s \right\|$$
$$\le \int_{\gamma_2} e^{\sigma t} || \Delta_{s,h}(A,B) || \, || g_1(s) || \,\mathrm{d}\sigma + \int_{\gamma_2} e^{\sigma t} || \Delta_{s,h}(A,B) || \, || x(0) || \, |s|^{\alpha-1}\,\mathrm{d}\sigma$$

if we let $r \downarrow 0$, and use the following inequalities (see equation below) we get (see (12)) Similarly, we can get the estimation of the integral of $e^{st} g_2(s)$ over γ_4 as

$$\lim_{r \downarrow 0} \left\| \int_{\gamma_4} e^{st} g_2(s)\,\mathrm{d}s \right\| \le \beta_2 ||B|| \, . \, ||\varphi|| e^{-hR^{\alpha}_{A,B}|d|} + \frac{\beta_2 ||x(0)||}{\alpha} |d|^{\alpha} \tag{13}$$

where $\beta_2 := \sup_{\sigma \in [d_0, 0]} ||(|\sigma|^{\alpha} e^{-i\pi\alpha} I - A - e^{-\sigma h} B)^{-1}||$. Next, we prove that

$$\lim_{r \downarrow 0} \int_{\gamma_2} e^{st} g_2(s)\,\mathrm{d}s = \int_d^0 e^{\sigma e^{i\pi} t} g_2(\sigma e^{i\pi}) d(\sigma e^{i\pi}) = \int_d^0 e^{\sigma e^{i\pi} t + i\pi} g_2(\sigma e^{i\pi})\,\mathrm{d}\sigma \tag{14}$$

In fact, for each $\varepsilon > 0$, we separate $\int_{\gamma_2} e^{st} g_2(s)\,\mathrm{d}s$ as follows:

$$\int_{\gamma_2} e^{st} g_2(s)\,\mathrm{d}s = \int_{s \in \gamma_2, 0 \ge \sigma \ge b_1} e^{st} g_2(s)\,\mathrm{d}s + \int_{s \in \gamma_2, d \le \sigma \le b_1} e^{st} g_2(s)\,\mathrm{d}s$$

where the number $b_1 \in (d, 0)$ is chosen such that $\beta_1 ||B|| \, ||\varphi|| e^{-hR^{\alpha}_{A,B}|b_1|} + \frac{\beta_1 ||x(0)||}{\alpha} |b_1|^{\alpha} \le \varepsilon/2$. By the same argument of the proof of (12), we have

$$\lim_{r \downarrow 0} \left\| \int_{s \in \gamma_2, 0 \ge \sigma \ge b_1} e^{st} g_2(s)\,\mathrm{d}s \right\| \le \varepsilon/2,$$
$$\left\| \int_{b_1}^0 e^{\sigma e^{i\pi} t} g_2(\sigma e^{i\pi}) d(\sigma e^{i\pi}) \right\| \le \varepsilon/2$$

Moreover, from Proposition 6 it follows that $\lim_{r \downarrow 0} \int_{s \in \gamma_2, d \le \sigma \le b_1} e^{st} g_2(s)\,\mathrm{d}s = \int_d^{b_1} e^{\sigma e^{i\pi} t} g_2(\sigma e^{i\pi})\mathrm{d}(\sigma e^{i\pi})$.

Therefore,
$\| \lim_{r \downarrow 0} \int_{\gamma_2} e^{st} g_2(s)\,\mathrm{d}s - \int_d^0 e^{\sigma e^{i\pi} t} g_2(\sigma e^{i\pi})\mathrm{d}(\sigma e^{i\pi}) \| \le \varepsilon$, which gives (14). Similarly, we can show (14) for the integral of $e^{st} g_2(s)$ over γ_4 as

$$\lim_{r \downarrow 0} \int_{\gamma_4} e^{st} g_2(s)\,\mathrm{d}s = -\int_d^0 e^{\sigma e^{-i\pi} t - i\pi} g_2(\sigma e^{-i\pi})\,\mathrm{d}\sigma \tag{15}$$

Step 4: *Estimation of* $\int e^{st} g_2(s)\,\mathrm{d}s$ *over* γ_1 *and* γ_5 *as* $r \downarrow 0$, $T \to +\infty$*:* From (8)–(11), (14) and (15), it follows the existence of the integral $\lim_{r \downarrow 0, T \to +\infty} (\int_{\gamma_1} + \int_{\gamma_5}) e^{st} g_2(s)\,\mathrm{d}s$, and hence the existence of the function $K(d, t)$ as (see (16)) Finally, using steps 1– 4 and the derived conditions (8)–(16), we can give an explicit form of the solution $x(t)$ as follows. From (9) it follows that

$$\lim_{r \downarrow 0, T \to +\infty} \frac{1}{2\pi i} \sum_{k=0}^{7} \int_{\gamma_k} e^{st} g_2(s)\,\mathrm{d}s = \lim_{r \downarrow 0, T \to +\infty} \oint_{\mathscr{D}} e^{st} g_2(s)\,\mathrm{d}s = 0$$

hence, using formula (8) we have (see equation below)
Assume that $d_0 \ge 0$. We consider the closed contour $\mathscr{D}^* = \gamma_0 \cup \gamma_6 \cup \gamma_7 \cup \gamma_8$ (see Fig. 2) where

$$\gamma_0 = \{s = \sigma + iT: d \le \sigma \le c\}, \quad \gamma_6 = \{s = \sigma - iT: d \le \sigma \le c\},$$
$$\gamma_7 = \{s = c + i\sigma: -T \le \sigma \le T\}, \quad \gamma_8 = \{s = d + i\sigma: -T \le \sigma \le T\}$$

Since $e^{st} g_2(s)$ is analytic in $\mathscr{D}^*$, we have $\oint_{\mathscr{D}^*} e^{st} g_2(s)\,\mathrm{d}s = 0$. Combining the equality, (8), (10) and (11) gives

$$x(t) = \lim_{T \to +\infty} \frac{1}{2\pi i} \int_{\gamma_7} e^{st} g_2(s)\,\mathrm{d}s = -\lim_{T \to +\infty} \frac{1}{2\pi i} \int_{\gamma_8} e^{st} g_2(s)\,\mathrm{d}s = \frac{1}{2\pi i} e^{\mathrm{d}t} K(d, t)$$

This completes the proof of theorem.

□

Remark 2: Note that the solution (6) can be rewritten in the following explicit form (see equation below) where (see equation below)

Remark 3: For the case of non-delayed linear FDEs, i.e. $B = 0$, we can verify that the formula (7) implies the standard solution of linear FDEs

$$||g_1(s)|| \le ||B|| \, ||\varphi|| e^{-dh} \le ||B|| \, . \, ||\varphi|| e^{-hR^{\alpha}_{A,B}}$$
$$\lim_{r \downarrow 0} \left[\sup_{s \in \gamma_2} ||\Delta_{s,h}(A,B)|| \right] \le \sup_{\sigma \in [d_0, 0]} ||(|\sigma|^{\alpha} e^{i\pi\alpha} I - A - e^{-\sigma h} B)^{-1}|| := \beta_1$$

$$\lim_{r \downarrow 0} \left\| \int_{\gamma_2} e^{st} g_2(s)\,\mathrm{d}s \right\| \le \beta_1 ||B|| \, ||\varphi|| e^{-hR^{\alpha}_{A,B}|d|} + \int_d^0 |\sigma|^{\alpha-1}\,\mathrm{d}\sigma \beta_1 ||x(0)|| = \beta_1 ||B|| \, ||\varphi|| e^{-hR^{\alpha}_{A,B}|d|} + \frac{\beta_1 ||x(0)||}{\alpha} |d|^{\alpha} \tag{12}$$

$$\lim_{r \downarrow 0, T \to +\infty} \left[\int_{\gamma_1} + \int_{\gamma_5} \right] e^{st} g_2(s)\,\mathrm{d}s = -e^{\mathrm{d}t} \lim_{r \downarrow 0, T \to \infty} \left[\int_r^T + \int_{-T}^{-r} \right] e^{i\sigma t} g_2(d + i\sigma) i\,\mathrm{d}\sigma = -e^{\mathrm{d}t} \int_{-\infty}^{\infty} e^{i\sigma t} g_2(d + i\sigma) i\,\mathrm{d}\sigma = -e^{\mathrm{d}t} K(d, t) \tag{16}$$

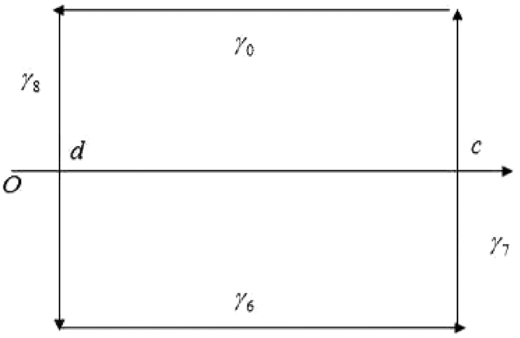

Fig. 2 *Boundary* $\mathscr{D}^*$

$$D^{\alpha}x(t) = Ax(t), \quad x(0) = x_0 \text{ as } x(t) = E_{\alpha}(At^{\alpha})x_0$$

In fact, using Propositions 1 and (7), we have

$$\begin{aligned} x(t) &= \frac{1}{2\pi i} e^{\mathrm{d}t} K(d,t) \\ &= \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} e^{st}[s^{\alpha} - A]^{-1} s^{\alpha-1} \, \mathrm{d}s \, x_0 \\ &= \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} e^{st} \sum_{k=0}^{\infty} \frac{A^k}{s^{\alpha(k+1)}} s^{\alpha-1} \, \mathrm{d}s \, x_0 \\ &= \sum_{k=0}^{\infty} \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} e^{st} \frac{1}{s^{\alpha k+1}} \, \mathrm{d}s A^k x_0 \\ &= \sum_{k=0}^{\infty} \frac{t^{\alpha k}}{\Gamma(\alpha k+1)} A^k x_0 = E_{\alpha}(At^{\alpha}) x_0 \end{aligned}$$

The proof of Theorem 2 does not only provide an explicit formula of the solution $x(t)$, but also a method to estimate the solution via the roots of the characteristic equations $\sigma_{A,B}^{\alpha,h}$. Next theorem gives an exponential bound for the solution of system (1). Before proving next theorem, we define some notations for simplicity

$$\beta_1 = \sup_{\sigma \in [d_0,0]} ||(|\sigma|^{\alpha} e^{i\pi\alpha} I - A - e^{-\sigma h} B)^{-1}||,$$

$$\beta_2 = \sup_{\sigma \in [d_0,0]} ||(|\sigma|^{\alpha} e^{-i\pi\alpha} I - A - e^{-\sigma h} B)^{-1}||,$$

$$k_2 = \frac{\beta_1 + \beta_2}{2\pi\alpha}||, \; k_1 = \frac{\beta_1 + \beta_2}{2\pi} e^{-hR_{A,B}^{\alpha}} ||B||,$$

$$\beta_3(d) = \sup_{-T_0 \le \sigma \le T_0} \| \Delta_{d+i\sigma,h}(A,B) \| ,$$

$$T_0 = \left[2||A|| + 2e^{-R_{A,B}^{\alpha}h}||B||\right]^{1/\alpha} + |R_{A,B}^{\alpha}|,$$

$$\begin{aligned} k_3(d) &= \frac{T_0 \beta_3(d) h}{\pi} e^{-R_{A,B}^{\alpha}h} \| B \| \\ &+ \frac{2 \| B \| [e^{-R_{A,B}^{\alpha}h} + 1 + he^{-R_{A,B}^{\alpha}h}] + 1}{\pi\alpha T_0^{\alpha}} + \frac{\beta_3(d) T_0^{\alpha}}{\pi\alpha} \\ &+ \frac{1}{2\pi}\left(\pi - 2\tan^{-1}\left(\frac{T_0}{|d|}\right) + 4\right) \end{aligned}$$

Theorem 3: Assume that $R_{A,B}^{\alpha} < d_0 < 0$. Then for all $d \in [d_0, 0]$ the solution of system (1) satisfies the following condition:

$$||x(t)|| \le (k_1|d| + k_2|d|^{\alpha} + k_3(d)e^{\mathrm{d}t}) \| \varphi \| , \quad t \ge 0 \qquad (17)$$

Proof: From the solution form (6) and using the derived estimations (12), (13) and (16), we can obtain the following estimation:

$$||x(t)|| \le k_1|d| \| \varphi \| + k_2|d|^{\alpha} \| \varphi \| + \frac{1}{2\pi}||K(d,t)||e^{\mathrm{d}t}$$

Thus, to obtain (17), we will estimate the value $\| K(d,t) \|$. Denoting $s = d + i\sigma$, we decompose the integral $K(d,t)$ as

$$K(d,t) = \int_{-\infty}^{+\infty} e^{i\sigma t} \Delta_{s,h}(A,B) g_1(s) i \, \mathrm{d}\sigma + \int_{-\infty}^{\infty} e^{i\sigma t} \Delta_{s,h}(A,B$$
$$)s^{\alpha-1} i \, \mathrm{d}\sigma x(0)$$

We will estimate the first integral: $J = \int_{-\infty}^{+\infty} e^{i\sigma t} \Delta_{s,h}(A,B) g_1(s) i \, \mathrm{d}\sigma$. Integrating of $g_1(s)$ by part gives (see equation below) hence for all $s = d + i\sigma$, $d \in [R_{A,B}^{\alpha}, 0]$ (see (18)) We now decompose the integral J as

$$\begin{aligned} x(t) &= \lim_{T \to +\infty} \frac{1}{2\pi i} \int_{\gamma_7} e^{st} g_2(s) \, \mathrm{d}s = -\lim_{r \downarrow 0, T \to +\infty} \frac{1}{2\pi i} \sum_{k=0}^{6} \int_{\gamma_k} e^{st} g_2(s) \, \mathrm{d}s \\ &= \frac{1}{2\pi i} \int_d^0 [e^{\sigma e^{-i\pi} t - i\pi} g_2(\sigma e^{-i\pi}) - e^{\sigma e^{i\pi} t + i\pi} g_2(\sigma e^{i\pi})] \, \mathrm{d}\sigma + \frac{1}{2\pi i} e^{\mathrm{d}t} K(d,t) \end{aligned}$$

$$\begin{aligned} x(t) &= X_1(t)\phi(0) + \int_d^0 \int_0^h X_2(t,\tau,\sigma) B\phi(\tau - h) \, \mathrm{d}\tau \, \mathrm{d}\sigma \\ &+ \int_{-\infty}^{\infty} \int_0^h X_3(t,\tau,\sigma) B\phi(\tau - h) \, \mathrm{d}\tau \, \mathrm{d}\sigma \end{aligned}$$

$$\begin{aligned} X_1(t) &= \frac{1}{2\pi i} \int_0^d [e^{\sigma e^{-i\pi} t - i\pi} \Delta_{\sigma e^{-i\pi},h}(A,B)(\sigma e^{-i\pi})^{\alpha-1} - e^{\sigma e^{-i\pi} t + i\pi} \Delta_{\sigma e^{i\pi},h}(A,B)(\sigma e^{i\pi})^{\alpha-1}] \, \mathrm{d}\sigma \\ &+ \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{(d+i\sigma)t} \Delta_{d+i\sigma,h}(A,B)(d+i\sigma)^{\alpha-1} i \, \mathrm{d}\sigma \\ X_2(t,\tau,\sigma) &= \frac{1}{2\pi i}\left[e^{\sigma e^{-i\pi}(t-\tau) - i\pi} \Delta_{\sigma e^{-i\pi},h}(A,B) - e^{\sigma e^{i\pi}(t-\tau) + i\pi} \Delta_{\sigma e^{i\pi},h}(A,B)\right] \\ X_3(t,\tau,\sigma) &= \frac{1}{2\pi i} e^{(d+i\sigma)(t-\tau)} \Delta_{d+i\sigma,h}(A,B) \end{aligned}$$

$$J=\int_{-T_0}^{T_0}+\int_{T_0}^{+\infty}+\int_{-\infty}^{-T_0}=J_1+J_2+J_3$$

From (18) and the continuity of $\Delta_{s,h}(A,B)$ w.r.t. σ on $[-T_0, 0]$ and $[0, T_0]$, it follows that

$$\| J_1 \| \le \int_{-T_0}^{T_0} \| \Delta_{s,h}(A,B) \| \, \| g_1(d+i\sigma) \| \, \mathrm{d}\sigma \le 2T_0\beta_3(d) \| B \| \, \| \varphi \| he^{-R^{\alpha}_{A,B}h}$$

Using the condition (18) and Proposition 2 gives

$$\| J_2 \| \le \| B \| \, \| \varphi \| [e^{-R^{\alpha}_{A,B}h}+1+he^{-R^{\alpha}_{A,B}h}]\int_{T_0}^{\infty}\frac{2}{|\sigma|^{\alpha}}\frac{1}{|\sigma|}\mathrm{d}\sigma=\frac{2\| B \| [e^{-R^{\alpha}_{A,B}h}+1+he^{-R^{\alpha}_{A,B}h}]}{\alpha T_0^{\alpha}}\| \varphi \|$$

because of $|\sigma|^{\alpha}\ge T_0^{\alpha}>2||A||+2e^{-R^{\alpha}_{A,B}h}||B||$. Similarly, we have

$$\| J_3 \| \le \frac{2\| B \| [e^{-R^{\alpha}_{A,B}h}+1+he^{-R^{\alpha}_{A,B}h}]}{\alpha T_0^{\alpha}}\| \varphi \|$$

Therefore (see (19)) We now are in position to estimate the second integral

$$\mathbb{J}=\int_{-\infty}^{\infty}e^{i\sigma t}\Delta_{s,h}(A,B)s^{\alpha-1}i\,\mathrm{d}\sigma x(0)$$

Similar to the estimation of integral J, we decompose the integral $\mathbb{J}$ into $\mathbb{J}_1, \mathbb{J}_2$ and $\mathbb{J}_3$ over the intervals $[-T_0, T_0]$, $[T_0, +\infty)$ and $(-\infty, -T_0]$, respectively. Setting $s=d+i\sigma$, we have

$$\| \mathbb{J}_1 \| \le \beta_3(d) \| x(0) \| \int_{-T_0}^{T_0}|\sigma|^{\alpha-1}\,\mathrm{d}\sigma=2\beta_3(d) \| x(0) \| \frac{T_0^{\alpha}}{\alpha} \quad (20)$$

Applying Proposition 6 gives

$$\mathbb{J}_2=\int_{T_0}^{\infty}\frac{e^{i\sigma t}}{s}i\,\mathrm{d}\sigma x(0)+\int_{T_0}^{\infty}e^{i\sigma t}0(s)\frac{1}{s^{\alpha+1}}x(0)i\,\mathrm{d}\sigma \quad (21)$$

$$\mathbb{J}_3=\int_{-\infty}^{-T_0}\frac{e^{i\sigma t}}{s}i\,\mathrm{d}\sigma x(0)+\int_{-\infty}^{-T_0}e^{i\sigma t}0(s)\frac{1}{s^{\alpha+1}}x(0)i\,\mathrm{d}\sigma \quad (22)$$

Using Proposition 6 again, the following inequalities hold:

$$\begin{aligned}\| \int_{T_0}^{\infty}\frac{e^{i\sigma t}0(s)}{s^{\alpha+1}}x(0)i\,\mathrm{d}\sigma \| &\le \int_{T_0}^{\infty}\frac{\| x(0) \| \, \| 0(s) \|}{\sigma^{\alpha+1}}\mathrm{d}\sigma\\ &\le \frac{(||A||+e^{-dh}||B||)\| x(0) \|}{(T_0^{\alpha}-||A||-e^{-dh}||B||)\alpha T_0^{\alpha}}\\ &\le \frac{T_0^{\alpha}/2}{T_0^{\alpha}-T_0^{\alpha}/2}\| \frac{\| x(0) \|}{\alpha T_0^{\alpha}}=\frac{\| x(0) \|}{\alpha T_0^{\alpha}}\end{aligned} \quad (23)$$

$$\begin{aligned}\| \int_{-\infty}^{-T_0}\frac{e^{i\sigma t}0(s)}{s^{\alpha+1}}x(0)i\,\mathrm{d}\sigma \| &\le \int_{-\infty}^{-T_0}\frac{\| 0(s) \| \, \| x(0) \|}{|\sigma|^{\alpha+1}}\mathrm{d}\sigma\\ &\le \frac{(||A||+e^{-dh}||B||)\| x(0) \|}{(T_0^{\alpha}-||A||-e^{-dh}||B||)\alpha T_0^{\alpha}}\\ &\le \frac{T_0^{\alpha}/2}{T_0^{\alpha}-T_0^{\alpha}/2}\frac{\| x(0) \|}{\alpha T_0^{\alpha}}=\frac{\| x(0) \|}{\alpha T_0^{\alpha}}\end{aligned} \quad (24)$$

because the function $u/(T_0^{\alpha}-u)$ is increasing w.r.t $u, 0\le u\le T_0^{\alpha}/2$ and $0\le ||A||+e^{-dh}||B||\le T_0^{\alpha}/2$. We now will estimate the integral $\int_{T_0}^{\infty}(e^{i\sigma t}/s)i\,\mathrm{d}\sigma+\int_{-\infty}^{-T_0}(e^{i\sigma t}/s)i\,\mathrm{d}\sigma$. We first note that

$$\int_{T_0}^{\infty}\frac{e^{i\sigma t}}{s}i\,\mathrm{d}\sigma+\int_{-\infty}^{-T_0}\frac{e^{i\sigma t}}{s}i\,\mathrm{d}\sigma=i\int_{T_0}^{\infty}\frac{\cos(\sigma t)2d}{d^2+\sigma^2}\mathrm{d}\sigma+\int_{T_0}^{\infty}\frac{\sin(\sigma t)2\sigma}{d^2+\sigma^2}\mathrm{d}\sigma \quad (25)$$

Then, we have

$$\begin{aligned}&\left|i\int_{T_0}^{\infty}\frac{\cos(\sigma t)2d}{d^2+\sigma^2}\mathrm{d}\sigma\right|\\ &\le \int_{T_0}^{\infty}\frac{|\cos(\sigma t)|2|d|}{d^2+\sigma^2}\mathrm{d}\sigma\\ &\le \int_{T_0}^{\infty}\frac{2|d|}{d^2+\sigma^2}\mathrm{d}\sigma=\pi-2\tan^{-1}\left(\frac{T_0}{|d|}\right)\end{aligned} \quad (26)$$

$$\begin{aligned}&\int_{T_0}^{\infty}\frac{\sin(\sigma t)2\sigma}{d^2+\sigma^2}\mathrm{d}\sigma\\ &=\sum_{k=k_0+1}^{\infty}(-1)^k\int_{k\pi}^{(k+1)\pi}\frac{\sin(u)2u}{(td)^2+u^2}du\\ &+\int_{tT_0}^{(k_0+1)\pi}\frac{\sin(u)2u}{(td)^2+u^2}du\end{aligned} \quad (27)$$

where k_0 satisfies $k_0\pi\le tT_0<(k_0+1)\pi$. By some simple calculations of Dirichlet complex integral and $T_0\ge |R^{\alpha}_{A,B}|>|d|$, we can estimate the integral (27) as follows: (see equation below) which gives

---

$$g_1(s)=\int_0^h e^{-s\tau}B\varphi(\tau-h)\,\mathrm{d}\tau=\frac{e^{-s\tau}}{-s}B\varphi(\tau-h)\Big|_{\tau=0}^{\tau=h}+\int_0^h\frac{e^{-s\tau}}{s}B\dot{\varphi}(\tau-h)\,\mathrm{d}\tau$$

$$\| sg_1(s) \| \le \| B \| \, \| \varphi \| [e^{-R^{\alpha}_{A,B}h}+1+he^{-R^{\alpha}_{A,B}h}],\ \| g_1(s) \| \le \| B \| \, \| \varphi \| he^{-R^{\alpha}_{A,B}h} \quad (18)$$

$$\| J \| \le 2T_0\beta_3(d) \| B \| \, \| \varphi \| he^{-R^{\alpha}_{A,B}h}+\frac{4\| B \| [e^{-R^{\alpha}_{A,B}h}+1+he^{-R^{\alpha}_{A,B}h}]}{\alpha T_0^{\alpha}}\| \varphi \| \quad (19)$$

$$\left|\int_{T_0}^{\infty}\frac{\sin(\sigma t)2\sigma}{d^2+\sigma^2}\mathrm{d}\sigma\right| \le \sum_{k=k_0+1}^{\infty}\frac{2}{k(k+1)}+2\le 4 \tag{28}$$

Hence, from the conditions (20)–(28), it follows that (see (29)) Owing to the derive conditions (19) and (29), we obtain $||K(d,t)|| \le \|J\| + \|\mathbb{J}\| \le 2\pi k_3(d)\|\varphi\|$, which finally gives

$$||x(t)|| \le (k_1|d| + k_2|d|^{\alpha} + k_3(d)e^{dt})\|\varphi\|$$

The proof of the theorem is completed. □

Remark 4: If $\alpha = 1$, we can use the similar argument of the proof of Theorem 2 for the case $\mathscr{D}^*$ to obtain $x(t) = (1/2\pi i)e^{dt}K(d,t)\|\varphi\|$. Then, according to [34], we have

$$||x(t)|| \le Ne^{dt}\|\varphi\|, \quad t \ge 0$$

which implies that the linear DE is exponentially stable in the Lyapunov sense due to $d < 0$.

In the sequel, we apply the obtained result to derive Lyapunov stability condition of linear FDDEs (1).

Theorem 4: If $R^{\alpha}_{A,B} < 0$, then the system (1) is asymptotically stable.

Proof: Since $R^{\alpha}_{A,B} < 0$, there is d_0 such that $R^{\alpha}_{A,B} < d_0 < 0$, we can apply Theorem 3 to get the exponential estimate (17), which immediately shows the Lyapunov stability of the system. It suffices to prove that $\lim_{t\to\infty} x(t) = 0$. Indeed, by Theorem 3 the following estimation holds for all $d \in [d_0, 0)$:

$$||x(t)|| \le (k_1|d| + k_2|d|^{\alpha} + k_3(d)e^{dt})\|\varphi\|, \quad t \ge 0$$

Note that the numbers k_1, k_2 do not depend on d, for every $\varepsilon > 0$, we can choose $d_1 \in (d_0, 0)$ such that

$$k_1|d_1|\,\|\varphi\| + k_2|d_1|^{\alpha}\|\varphi\| \le \frac{\varepsilon}{2}$$

On the other hand, since $d_1 < 0$, there exists $T > 0$ such that

$$k_3(d_1)e^{d_1 t}\|\varphi\| \le \frac{\varepsilon}{2}, \quad \forall t > T$$

Therefore, for all $t > T$ we obtain $\|x(t)\| \le \varepsilon$, which implies that $\lim_{t\to\infty} x(t) = 0$. This completes the proof of the theorem.
□

Remark 5: For the case $A = 0$, similar stability conditions were given in [12–14] by using singular value decomposition of the characteristic equations. Moreover, it is worth noting that the asymptotic stability of the LFDDEs (1) can be verified by the Lyapunov stability theorem (Theorem 4.1 in [19] or Theorem 4 in [22]) using Lyapunov function method. However, the solution leads to solving LMIs depending either on the trace of the system matrices or on the positive definite matrix solution, which is not easy to solve and to verify the Lyapunov stability conditions (see, e.g. Example 1).

Next theorem gives a sufficient condition on the FTS of system (1), which is less restrictive than the condition obtained in [28, 31–33].

Theorem 5: The system (1) is finite-time stable w.r.t (c_1, c_2, T) if

$$E_{\alpha}([||A|| + ||B||]T^{\alpha}) \le \frac{c_2}{c_1} \tag{30}$$

Proof: As in the proof of Theorem 1, we have derived the condition (4) as

$$||x(t)|| \le \|x_t\| \le ||\varphi||E_{\alpha}([||A|| + ||B||]T^{\alpha})$$

Therefore, if $\|\varphi\| \le c_1$ and due to the assumption (30) we then have

$$\|x(t)\| \le c_2, \quad \forall t \in [0, T]$$

This completes the proof of the theorem. □

Remark 6: We note that by using the estimation (5) on the function $E_{\alpha}(t)$, the sufficient condition (30) can be relaxed by the following condition:

$$\begin{aligned}\exists \eta > 0: \quad & E_{\alpha}(t) \le \frac{\eta}{\alpha}\exp\{t^{1/\alpha}\}, \quad \forall t \ge 0,\\ & \frac{\eta}{\alpha}\exp\{(\|A\| + \|B\|)^{1/\alpha}T\} \le \frac{c_2}{c_1}\end{aligned} \tag{31}$$

Remark 7: In Theorem 5 by considering a special case of LFDDEs (1) (i.e. $A = 0$) and under the strict assumption on the initial delay function $\varphi(t) \in C^1([-h, 0], R^n)$, Li and Wang [28] proposed a similar FTS condition. The different method used in [28] is an extension of the Mittag–Leffler function, which allows a result on FTS to be derived via the delayed Mittag–Leffler type matrix function.

The following examples are given to illustrate the validity and effectiveness of the proposed stability results.

Example 1 (asymptotic stability): Consider system (1), where

$$A = \begin{bmatrix}0 & -2\\ 2 & 0\end{bmatrix}, \quad B = \begin{bmatrix}0.1 & 0.1\\ 0.1 & 0.1\end{bmatrix},$$
$$\alpha = \frac{1}{2}, \quad \varphi(t) = [1 \quad 1]^{\top}, \quad t \in [-2, 0]$$

We show that $R^{1/2}_{A,B} < 0$. Indeed, we have for all $s \in C$, $\mathrm{Re}(s) \ge 0$

$$\det(s^{1/2}I - A) = \det\begin{bmatrix}s^{1/2} & 2\\ -2 & s^{1/2}\end{bmatrix} = s + 4 \ne 0$$

which shows that the matrix $(s^{1/2}I - A)$ is invertible and

$$\left|\sum_{k=k_0+1}^{\infty}(-1)^k\int_{k\pi}^{(k+1)\pi}\frac{\sin(u)2u}{(td)^2+u^2}\mathrm{d}u\right| \le \sum_{k=k_0+1}^{\infty}\frac{2}{k(k+1)}$$
$$\left|\int_{tT_0}^{(k_0+1)\pi}\frac{\sin(u)2u}{(td)^2+u^2}\mathrm{d}u\right| \le \left|\int_{k_0\pi}^{(k_0+1)\pi}\frac{\sin(u)2u}{(td)^2+u^2}\mathrm{d}u\right| \le 2\int_0^{\pi}\frac{\sin(u_1)}{u_1}\mathrm{d}u_1 \le 2$$

$$\|\mathbb{J}\| \le 2\beta_3(d)\frac{T_0^{\alpha}}{\alpha}\|x(0)\| + 2\frac{1}{\alpha T_0^{\alpha}}\|x(0)\| + \left(\pi - 2\tan^{-1}\left(\frac{T_0}{|d|}\right) + 4\right)\|x(0)\| \tag{29}$$

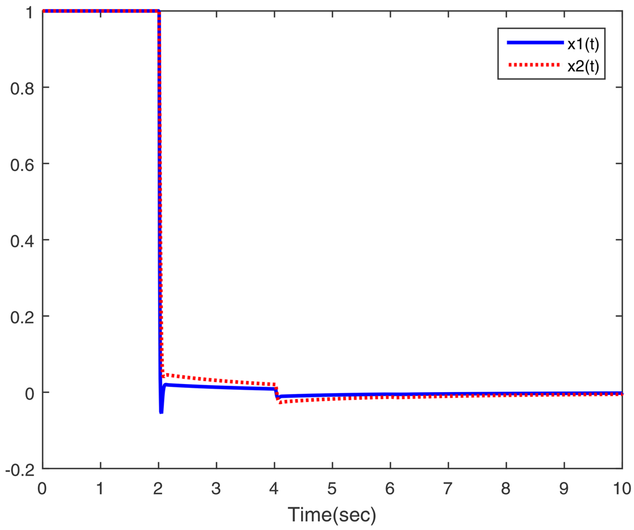

Fig. 3 *State response of the system in Example 1*

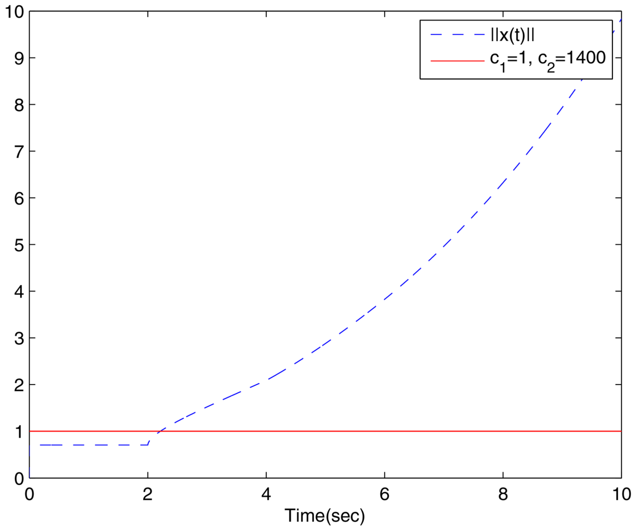

Fig. 4 *State response of the system in Example 2*

$$(s^{1/2}I - A)^{-1} = \frac{1}{s+4}\begin{bmatrix} s^{1/2} & 2 \\ -2 & s^{1/2}\end{bmatrix}$$

$$\overline{[(s^{1/2}I - A)^{-1}]^{\mathrm{T}}}(s^{1/2}I - A)^{-1} = \frac{1}{|s+4|^2}$$

$$\begin{bmatrix} |s|+4 & 2(\bar{s}^{1/2} - s^{1/2}) \\ -2(\bar{s}^{1/2} - s^{1/2}) & |s|+4 \end{bmatrix}$$

The eigenvalues of $\overline{[(s^{1/2}I - A)^{-1}]^{\mathrm{T}}}(s^{1/2}I - A)^{-1}$ are defined as

$$\lambda = \frac{|s|+4}{|s+4|^2} \pm 2i\frac{\bar{s}^{1/2} - s^{1/2}}{|s+4|^2} \le \frac{|s|+4}{|s+4|^2} + 4\frac{\sqrt{|s|}}{|s+4|^2} \le 2 + 4 = 6$$

Therefore, we have

$$\| (s^{1/2}I - A)^{-1} \| \le \sqrt{6}, \quad \forall s \in C, \quad \mathrm{Re}(s) \ge 0$$

Moreover, the matrix $I + (s^{1/2}I - A)^{-1}Be^{-2s}$ is invertible for $\mathrm{Re}(s) \ge 0$ because of

$$\|(s^{1/2}I - A)^{-1}Be^{-2s}\| \le \|(s^{1/2}I - A)^{-1}\| \|B\| \le \sqrt{6} \times 0.2 < 1$$

On the other hand, since the matrix

$$s^{1/2}I - A - Be^{-2s} = (s^{1/2}I - A)\left(I + (s^{1/2}I - A)^{-1}Be^{-2s}\right)$$

is invertible, we have $R_{A,B}^{1/2} < 0$. The system, by Theorem 4, is asymptotically stable. Fig. 3 shows the trajectories of $x_1(t)$ and $x_2(t)$ of the system with the initial condition $\varphi(t) = (1, 1), t \in [-2, 0]$. It is worth noting that the asymptotic stability of the system cannot be verified by using the Lyapunov function method, i.e. there is no Lyapunov functional applied to the system. In fact, if any Lyapunov functional is applied to the system (e.g. by Theorem 4.1 in [19] or by Theorem 4 in [22]), then the stability condition leads to an LMI of the form

$$\begin{bmatrix} AP + PA^{\mathrm{T}} + qP & BP \\ PB^{\mathrm{T}} & -I \end{bmatrix} < 0$$

where $q > 1, P$ is a symmetric positive definite matrix. By the Schur complement lemma, this LMI implies $A^{\mathrm{T}}P + PA < 0$, and hence by the Lyapunov equation stability theorem, matrix A is Hurwitz. However, it is obvious that the matrix A in the example is not Hurwitz because of the real part of its eigenvalues ($\lambda(A) = \pm i\sqrt{2}$) is zero.

Example 2 (FTS): Consider system (1), where

$$A = \begin{bmatrix} 0.5 & -\frac{1}{6} \\ 0.25 & \frac{1}{3} \end{bmatrix}, \quad B = \begin{bmatrix} 0.25 & 0 \\ 0 & 0.25 \end{bmatrix},$$

$$\alpha = \frac{1}{2}, \quad \varphi(t) = [0.5 \quad 0.5]^{\mathrm{T}}, \quad t \in [-2, 0]$$

We have $\| A \| = 0.25\sqrt{5}$, $\| B \| = 0.25$, and it is easy to verify the validity of the condition (30) or (31) for $c_1 = 1, c_2 = 1400, T = 10$. For example, the condition (30) holds by calculating the value of $E_\alpha(t)$ by using the following formula:

$$E_{1/2}(z) = \frac{2}{\sqrt{\pi}}e^{z^2}\int_{-z}^{\infty} e^{-u^2}\mathrm{d}u, \quad z \in R$$

which gives $E_{1/2}([0.25\sqrt{5} + 0.25]\sqrt{10}) = 1391.4$.

Therefore, the system, by Theorem 5, is finite-time stable w.r.t. $(1, 1400, 10)$. Moreover, we can show that this system cannot be finite-time stable by using the conditions obtained in [31–33]. Indeed, for example, by [31] the system is FTS w.r.t $c_1 = 1, c_2 = 1400, T = 10$, if

$$\left[1 + \frac{[\lambda_{\max}(A) + \lambda_{\max}(B)]T^{\alpha}}{\Gamma(\alpha+1)}\right]\exp\left\{\frac{[\lambda_{\max}(A) + \lambda_{\max}(B)]T^{\alpha}}{\Gamma(\alpha+1)}\right\} \le \frac{c_2}{c_1}$$

For $\alpha = 1/2$, we have

$$\Gamma(\alpha+1) = \frac{\sqrt{\pi}\Gamma(2+1)}{2^{2^n}\Gamma(1+1)} = \frac{\sqrt{\pi}(2)!}{2^2 1!} = 0.886$$

Therefore, we have

$$\left[1 + \frac{12.921\sqrt{10}}{0.886}\right]\exp\left\{\frac{12.921\sqrt{10}}{0.886}\right\} > \frac{1400}{1}$$

which implies that the system is not finite-time stable w.r.t. $c_1 = 1, c_2 = 1400$ and $T = 10$. Fig. 4 shows the trajectories of $\| x(t) \|$ of the system with the initial condition $\varphi(t) = (0.5, 0.5), t \in [-2, 0]$.

4 Conclusion

In this paper, we have studied the stability of LFDDEs. The proposed analytical tools used in the proof are based on the Laplace transform method, Mittag–Leffler function and the generalised Gronwall inequality. This approach has a wider usage and can be applied to derive an explicit formula of solutions of

LFDDEs, precise exponential estimates for the solutions, sufficient conditions for asymptotic stability and FTS. Finally, illustrative examples for the proposed results have been presented.

5 Acknowledgments

This work was supported by the School of Engineering Strategic Fund, Deakin University, Australia and the research of Vu N. Phat was supported by the project VAST.HTQT.Php.03/16-17 between VAST and CNRS.

6 References

[1] Kiryakova, V.: '*Generalized fractional calculus and applications*' (Longman Group UK Limited, Essex, UK, 1994)

[2] Podlubny, I.: '*Fractional differential equations*' (Academic Press, San Diego, 1999)

[3] Kilbas, A.A., Srivastava, H., Trujillo, J.: '*Theory and applications of fractional differential equations*' (Elsevier Science, Amsterdam, 2006)

[4] Wang, J., Zhou, Y., Feckan, M.: 'Nonlinear impulsive problems for fractional differential equations and Ulam stability', *Comput. Math. Appl.*, 2012, **64**, pp. 3389–3405

[5] Yang, H., Jiang, B.: 'Stability of fractional-order switched non-linear systems', *IET Control Theory Appl.*, 2016, **10**, pp. 965–970

[6] Ge, F., Kou, C.: 'Stability analysis by Krasnoselskii fixed point theorem for nonlinear fractional differential equations', *Appl. Math. Comput.*, 2015, **257**, pp. 308–316

[7] Ye, H., Gao, J., Ding, Y.: 'A generalized Gronwall inequality and its application to a fractional differential equation', *J. Math. Anal. Appl.*, 2007, **328**, pp. 1075–1081

[8] Lazarevi, M.P., Spasi, A.M.: 'Finite-time stability analysis of fractional order time-delay systems: Gronwall's approach', *Math. Comput. Model.*, 2009, **49**, pp. 475–481

[9] Deng, W., Li, C., Lu, J.: 'Stability analysis of linear fractional differential system with multiple time delays', *Nonlinear Dyn.*, 2007, **48**, pp. 409–416

[10] Johnson, M.A.: 'Stability of small periodic waves in fractional KdV-type equations', *SIAM J. Math. Anal.*, 2013, **45**, pp. 3168–3193

[11] Chen, Y., Moore, K.L.: 'Analytical stability bound for a class of delayed fractional-order dynamic systems', *Nonlinear Dyn.*, 2002, **29**, pp. 191–200

[12] De la Sen, M.: 'About robust stability of Caputo linear fractional dynamic systems with time delays through fixed point theory', *Fixed Point Theory Appl.*, 2011, **2011**, ID 867932, pp. 1–19

[13] Li, C.P., Zhang, F.R.: 'A survey on the stability of fractional differential equations', *Eur. Phys. J., Spec. Top.*, 2011, **193**, pp. 27–47

[14] Cermak, J., Hornicek, J., Kisela, T.: 'Stability regions for fractional differential systems with a time delay', *Commun. Nonlinear Sci. Numer. Simul.*, 2016, **31**, pp. 108–123

[15] Kaslik, E., Sivasundaram, S.: 'Analytical and numerical methods for the stability analysis of linear fractional delay differential equations', *J. Comput. Appl. Math.*, 2012, **236**, pp. 4027–4041

[16] Hale, J.K., Verduyn Lunel, S.M.: '*Introduction to functional differential equations*' (Springer Science-Business Media, New York, 1993)

[17] Erdelyi, A.: '*Higher transcendental functions*' (McGraw-Hill, New York, 1953)

[18] Li, Y., Chen, Y.Q., Podlubny, I.: 'Stability of fractional-order nonlinear dynamic systems: Lyapunov direct method and generalized Mittag–Lefflers stability', *Comput. Math. Appl.*, 2010, **59**, pp. 1810–1821

[19] Baleanu, D., Sadati, S.J., Ghaderi, R., *et al.*: 'Razumikhin stability theorem for fractional systems with delay', *Abs. Appl. Anal.*, 2010, **2010**, ID 124812, pp. 1–9

[20] Duarte-Mermoud, M.A., Aguila-Camacho, N., Gallegos, J.A., *et al.*: 'Using general quadratic Lyapunov functions to prove Lyapunov uniform stability for fractional order systems', *Commun. Nonlinear Sci. Numer. Simul.*, 2015, **22**, pp. 650–659

[21] Sabatier, J., Moze, M., Farges, C.: 'LMI stability conditions for fractional order systems', *Comput. Math. Appl.*, 2010, **59**, pp. 1594–1609

[22] Wen, Y., Zhou, X.F., Zhang, Z., *et al.*: 'Lyapunov method for nonlinear fractional differential systems with delay', *Nonlinear Dyn.*, 2015, **74**, pp. 1015–1025

[23] Gao, Z., Liao, X.: 'Robust stability criterion of fractional-order functions for interval fractional-order systems', *IET Control Theory Appl.*, 2013, **7**, pp. 60–67

[24] Agarwal, R., O'Regan, D., Hristova, S., *et al.*: 'Practical stability with respect to initial time difference for Caputo fractional differential equations', *Commun. Nonlinear Sci. Numer. Simul.*, 2017, **42**, pp. 106–120

[25] Li, K., Peng, J.: 'Laplace transform and fractional differential equations', *Appl. Math. Lett.*, 2011, **24**, pp. 2019–2023

[26] Wang, J., Zhang, Y.: 'On the concept and existence of solutions for fractional impulsive systems with Hadamard derivatives', *Appl. Math. Lett.*, 2015, **39**, pp. 85–90

[27] Wang, J., Ibrahim, A.G., l Feckan, M.: 'Nonlocal impulsive fractional differential inclusions with fractional sectorial operators on Banach spaces', *Appl. Math. Comput.*, 2015, **257**, pp. 103–118

[28] Li, M., Wang, J.: 'Finite time stability of fractional delay differential equations', *Appl. Math. Lett.*, 2017, **64**, pp. 170–176

[29] Krol, K.: 'Asymptotic properties of fractional delay differential equations', *Appl. Math. Comput.*, 2011, **218**, pp. 1515–1532

[30] Yang, Z.Z., Cao, J.: 'Initial value problems for arbitrary order fractional differential equations with delay', *Commun. Nonlinear Sci. Numer. Simul.*, 2013, **18**, pp. 2993–3005

[31] Lazarevic, M., Debeljkovic, D.: 'Finite-time stability analysis of linear autonomous fractional order systems with delayed state', *Asian J. Control*, 2005, **7**, pp. 440–447

[32] Rakkiyappan, R., Velmurugan, G., Cao, J.: 'Finite-time stability analysis of fractional-order complex-valued memristor-based neural networks with time delays', *Nonlinear Dyn.*, 2014, **78**, pp. 2823–2836

[33] Hei, X., Wu, R.: 'Finite-time stability of impulsive fractional-order systems with time-delay', *Appl. Math. Model.*, 2016, **40**, pp. 4285–4290

[34] Pachpatte, B.G.: '*Inequalities for differential and integral equations*' (Academic Press, New York, 1998)