Full-Order Observer Design for Nonlinear Complex Large-Scale Systems with Unknown Time-Varying Delayed Interactions

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This article is concerned with the problem of state observer for complex large-scale systems with unknown time-varying delayed interactions. The class of large-scale interconnected systems under consideration is subjected to interval time-varying delays and nonlinear perturbations. By introducing a set of argumented Lyapunov–Krasovskii functionals and using a new bounding estimation technique, novel delay-dependent conditions for existence of state observers with guaranteed exponential stability are derived in terms of linear matrix inequalities (LMIs). In our design approach, the set of full-order Luenberger-type state observers are systematically derived via the use of an efficient LMI-based algorithm. Numerical examples are given to illustrate the effectiveness of the result. © 2014 Wiley Periodicals, Inc. Complexity 21: 123–133, 2015

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1. INTRODUCTION

Stability analysis of large-scale interconnected systems has been the subject of considerable research attention in the literature (see, for example [1–3]). However, the problem of designing decentralized state observers for nonlinear large-scale interconnected delay systems still faces many challenges; particularly, when the measurement of all the states is not available and the inevitable presences of time-varying delays and nonlinear perturbations in the systems. When the knowledge of the states is not available, a state observer is designed to provide vital information of the system and an observer-based feedback control scheme can be realized [4–8]. Nevertheless, for large-scale interconnected systems which are under the constraint of decentralized information

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(i.e., interchange of information among the subsystems is not allowed) and also subjected to time-varying delays and nonlinear perturbations, the problem of designing decentralized state observers still has not been adequately addressed. Conversely, when there are time-varying delays presented in the systems, they can be modeled in the states, outputs and the interconnections. However, very often, an exact real-time knowledge of the time-varying delay is not known or available. Furthermore, it is more realistic and practical to model the delay varies within an interval, with specified lower and upper bounds, in which the lower bound is not restricted to be zero and the time derivatives of the delay be allowed to be undefined or unknown. However, most existing work on the state observer of time-delay interconnected systems either assumes that the time-delay is a known constant or differentiable with boundedness of its time derivatives. There have been some papers addressing the problem of stability and state observer of complex large-scale systems with time-varying delays [9–14]. Nevertheless, the practical and theoretical issues stem from the unavailability of time delays and nonlinear perturbations have not yet been addressed in the literature. It is worthwhile to note that these issues are in fact quite complicated to solve, and therefore, there is a strong need for more research in the design of full-order state observer for nonlinear interconnected systems with interval time-varying delays.

This article considers a general class of complex large-scale systems where time-varying delays and nonlinear purturbations are presented in both the state and the observation output. Under the practical constraint of decentralized information coupled with the facts that the measurement of all the states and the real-time knowledge of the interval time-varying delays are not available, our objective is to design a set of state observers to exponentially stabilize the error system with a decay rate of convergence. Here, due to the constraint that interchange of information among the subsystems is not possible and the real-time knowledge of the interval time-varying delays is not available, we, therefore, have to use a set of completely memoryless decentralized full-order Luenberger-type state observers. The perturbations presented in the states and observation outputs are described by nonlinear functions satisfying the Lipschitzian condition. To solve the problem posed in this article, we introduce a set of augmented Lyapunov–Krasovskii functionals associated with the lower and upper bounds of the time delays. With this new set of Lyapunov–Krasovskii functionals and the new bounding estimation technique we derive new delay-dependent LMI stabilizability conditions for exponential stability of the error system.

The article is organized as follows. Section 2 presents the problem statement together with definitions and some well-known technical propositions needed for the proof of the main result. State observer design for exponential stability with numerical examples showing the effectiveness of the proposed method is presented in Section 3.

2. PRELIMINARIES

The following notations will be used throughout this article. R^+ denotes the set of all real positive numbers; R^n denotes the n–dimensional space; $R^{n\times r}$ denotes the space of all $(n\times r)$-matrices. The notation $i=\overline{1,N}$ means $i=1,2,\ldots,N$; A^{T} denotes the transpose of A; a matrix A is symmetric if $A=A^{\mathrm{T}}$; I denotes the identity matrix; $\lambda(A)$ denotes the set of all eigenvalues of A; $\lambda_{\max}(A)=\max\{\mathrm{Re}\lambda : \lambda\in\lambda(A)\}$; $\lambda_{\min}(A)=\min\{\mathrm{Re}\lambda : \lambda\in\lambda(A)\}$; $\lambda_A=\lambda_{\max}(A^{\mathrm{T}}A)$; $C^1([a,b],R^n)$ denotes the set of all R^n-valued differentiable functions on $[a,b]$; $L_2([0,\infty],R^r)$ stands for the set of all square-integrable R^r-valued functions on $[0,\infty)$. The symmetric terms in a matrix are denoted by *. Matrix A is semi-positive definite ($A\geq 0$) if $(Ax,x)\geq 0$, for all $x\in R^n$; A is positive definite ($A>0$) if $(Ax,x)>0$ for all $x\neq 0; A\geq B$ means $A-B\geq 0$. The segment of the trajectory $x(t)$ is denoted by $x_t=\{x(t+s) : s\in[-\tau,0]\}$ with its norm $||x_t||=\sup_{s\in[-\tau,0]}||x(t+s)||$.

Consider a class of large-scale nonlinear systems which can be usually characterized by a large number of variables representing the system, a strong interaction between subsystem variables, and a complex interaction between subsystems [1,2] described by the following equation:

$$\begin{cases} \dot{x}_i(t)=A_ix_i(t)+\sum_{j\neq i,j=1}^{N}A_{ij}x_j(t-h_{ij}(t))+f_i(t,x_i(t),\{x_j(t-h_{ij}(t))\}_{j=1,j\neq i}^{N}), \\ z_i(t)=C_ix_i(t)+g_i(t,x_i(t)), t\geq 0, \end{cases} \tag{2.1}$$

with the initial conditions

$$x_i(t_0+\theta)=\varphi_i(\theta),\quad \forall\theta\in\mathcal{I}_{t_0,h},\quad (t_0,\varphi_i)\in R^+\times C([-\tau,0],R^{n_i}),$$

where $\varphi_i:\mathcal{I}_{t_0,h}\to R^{n_i}$ is a continuous norm-bounded initial condition (see also [13]) and $\mathcal{I}_{t_0,h}=\{t\in R : t=\eta-h(\eta)\leq t_0, \eta\geq t_0\}$; $\tau=\sup_{t_0\in R^+,t\in\mathcal{I}_{t_0,h}}(t_0-t)$;

$h_{ij}(t):R^+\to R^+$ is a continuous function satisfying

$$0\leq h_1\leq h_{ij}(t)\leq h_2,\quad t\geq 0,\forall i,j=\overline{1,N},$$

where h_1 and h_2 is given real non-negative numbers and $h_1\neq h_2$. We see in this case that $h_2=\tau; x^{\mathrm{T}}(t)=[x_1(t)^{\mathrm{T}},\ldots,x_N(t)^{\mathrm{T}}], x_i(t)\in R^{n_i}$ is the state vector, $z_i(t)\in R^{q_i}$ is the output vector. The systems matrices A_i,C_i,A_{ij} are of appropriate dimensions; the nonlinear functions $f_i(\cdot)$ and $g_i(\cdot)$ satisfy the following conditions

$$\exists a_i,a_{ij}>0 : ||f_i(t,x_i(t),\{x_j(t-h_{ij}(t))\}_{j=1,j\neq i}^{N})||\leq a_i||x_i(t)|| + \sum_{j\neq i,j=1}^{N}a_{ij}||x_j(t-h_{ij}(t))||$$

$$\exists g_i>0 : ||g_i(t,y_1)-g_i(t,y_2)||\leq g_i||y_1-y_2||,\forall y_1,y_2\in R^{n_i},t\in R^+. \tag{2.2}$$

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We assume $\varphi_i(.) \in C^1([-h_2,0],R^{n_i})$ and $||\varphi_i||_{C_1}=\sup_{t\in[-h_2,0]}||\varphi_i(t)||+\sup_{t\in[-h_2,0]}||\dot{\varphi}_i(t)||$ stands for the norm of a function $\varphi_i(.) \in C^1([-h_2,0],R^{n_i})$. Once the above assumption on $\varphi_i(.), f_i(.), g_i(.)$ are given, the solution of system (2.1) is well defined (see, e.g., [15]).

Due to the fact that not all of the state variables are available for state observer purpose and that the real-time knowledge of the delay, $h_{ij}(t)$, is not available, we, therefore, consider the following decentralized full-order Luenberger state observer for the system (2.1):

$$\begin{aligned}&\dot{\hat{x}}_i(t)=A_i\hat{x}_i(t)+L_i[z_i(t)-C_i\hat{x}_i(t)-g_i(t,\hat{x}_i(t))], \quad t\geq 0,\\&\hat{x}_i(0)=0, \quad i=\overline{1,N},\end{aligned} \tag{2.3}$$

in which $\hat{x}_i(t)$ is the observer state vector of the $i-$th subsystem, $L_i \in \mathbb{R}^{n_i\times q_i}$ is the observer gain matrices to be designed.

Define an error vector $e_i(t)=x_i(t)-\hat{x}_i(t), i=\overline{1,N}$, which denotes the difference between the real state and the estimated state vector of the $i-$th subsystem. Then, we have the following error system

$$\begin{aligned}\dot{e}_i(t)=&A_ie_i(t)-L_i[z_i(t)-C_i\hat{x}_i(t)-g_i(t,\hat{x}_i(t))]+\sum_{j\neq i,j=1}^{N}A_{ij}x_j(t-h_{ij}(t))\\&+f_i(t,x_i(t),\{x_j(t-h_{ij}(t))\}_{j=1,j\neq i}^{N}).\end{aligned} \tag{2.4}$$

It is clear from (2.4) that the error system is rather complex and certainly the task of stabilizing (2.4) is not an easy and trivial task. In this article, the problem to be addressed is to systematically derive the observer gain matrix $L_i, i=\overline{1,N}$, so that the error system (2.4) is exponentially stable with a prescribed β-convergence rate. Let us now recall the following definitions (see, e.g., [4]) and propositions that will be used to derive the main results of the article.

Definition 2.1.

Given $\beta>0$. The error system (2.4) *is β-stable if there is positive number $N_0>0$ such that every solution of the system satisfies:*

$$||e(t,\varphi)||\leq N_0||\varphi||_h e^{-\beta t}, \quad \forall t\geq 0,$$

where $||\varphi||_h=\sqrt{\sum_{i=1}^{N}||\varphi_i||_{C_1}^2}$.

Proposition 2.1 (Schur complement lemma [16])

Given matrices X, Y, Z, where $Y=Y^{\mathrm{T}}>0, X=X^{\mathrm{T}}$. Then, $X+Z^{\mathrm{T}}Y^{-1}Z<0$ if and only if

$$\begin{bmatrix}X & Z^{\mathrm{T}}\\ Z & -Y\end{bmatrix}<0.$$

Proposition 2.2. (Jensen-type integral inequality [17])

For any constant matrix $Z=Z^{\mathrm{T}}>0$ and scalar $h,\bar{h},0<h<\bar{h}$ such that the following integrations are well defined, then

$$-\int_{t-h}^{t}x(s)^{\mathrm{T}}Zx(s)ds\leq-\frac{1}{h}\left(\int_{t-h}^{t}x(s)ds\right)^{\mathrm{T}}\times Z\left(\int_{t-h}^{t}x(s)ds\right).$$

$$-\int_{-\bar{h}}^{-h}\int_{t+s}^{t}x(\tau)^{\mathrm{T}}Zx(\tau)d\tau ds\leq-\frac{2}{\bar{h}^2-h^2}\left(\int_{-\bar{h}}^{-h}\int_{t+s}^{t}x(\tau)d\tau ds\right)^{\mathrm{T}}\times Z\left(\int_{-\bar{h}}^{-h}\int_{t+s}^{t}x(\tau)d\tau ds\right).$$

Proposition 2.3. (Lower bounds lemma [18])

Let $f_1,f_2,\dots,f_N:R^m\to R$ have positive values in an open subset D of R^m. Then, the reciprocally convex combination of f_i over D satisfies

$$\min_{\{r_i|r_i>0,\sum_i r_i=1\}}\sum_i\frac{1}{r_i}f_i(t)=\sum_i f_i(t)+\max_{g_{i,j}(t)}\sum_{i\neq j}g_{i,j}(t)$$

subject to

$$\left\{g_{i,j}:R^m\to R, g_{j,i}(t)=g_{i,j}(t), \begin{bmatrix}f_i(t) & g_{i,j}(t)\\ g_{i,j}(t) & f_j(t)\end{bmatrix}\geq 0\right\}.$$

Proposition 2.4. (Cauchy matrix inequality [16])

For any $x,y\in R^n$ and positive definite matrix $M\in R^{n\times n}$, we have

$$2x^{\mathrm{T}}y\leq y^{\mathrm{T}}My+x^{\mathrm{T}}M^{-1}x.$$

3. DESIGN OF FULL-ORDER LUENBERGER-TYPE OBSERVER

In this section, we give a design of the full-order state observer for nonlinear system (2.1) such that the error of system (2.4) is exponentially stable. Before introducing the main result, the following notations of several matrix variables are defined for simplicity.

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$$f_i(\cdot)=f_i(t,x_i(t),\{x_j(t-h_{ij}(t))\}_{j=1,j\neq i}^{N}),\ \xi_i=a_i+\sum_{j\neq i,j=1}^{N}a_{ij},$$

$$H^i_{i,i}=P_iA_i+A_i^{\mathrm{T}}P_i+2\beta P_i+2Q_i-e^{-2\beta h_1}R_i-e^{-2\beta h_2}R_i-\frac{2e^{-4\beta h_2}(h_2-h_1)}{h_2+h_1}\Lambda_i+3a_iI,$$

$$H^i_{i,N+1}=e^{-2\beta h_1}R_i,H^i_{i,N+2}=e^{-2\beta h_2}R_i,H^i_{i,N+3}=A_i^{\mathrm{T}}P_i,\quad H^i_{i,N+4}=\frac{2e^{-4\beta h_2}}{h_2+h_1}\Lambda_i,$$

$$H^i_{j,j}=(3a_{ji}+3)I-\frac{2e^{-2\beta h_2}}{N-1}U_i+\frac{e^{-2\beta h_2}}{N-1}(S_i+S_i^{\mathrm{T}}),\quad j\neq i,j=\overline{1,N},$$

$$H^i_{j,(N+1)}=\frac{e^{-2\beta h_2}}{N-1}U_i-\frac{e^{-2\beta h_2}}{N-1}S_i,H^i_{j,(N+2)}=\frac{e^{-2\beta h_2}}{N-1}U_i-\frac{e^{-2\beta h_2}}{N-1}S_i^{\mathrm{T}},\quad j\neq i,j=\overline{1,N},$$

$$H^i_{N+1,N+1}=-e^{-2\beta h_1}Q_i-e^{-2\beta h_1}R_i-e^{-2\beta h_2}U_i,\quad H^i_{N+1,N+2}=e^{-2\beta h_2}S_i^{\mathrm{T}},$$

$$H^i_{N+2,N+2}=-e^{-2\beta h_2}Q_i-e^{-2\beta h_2}R_i-e^{-2\beta h_2}U_i,$$

$$H^i_{N+3,N+3}=(h_1^2+h_2^2)R_i+(h_2-h_1)^2U_i+(h_2-h_1)h_2\Lambda_i-2P_i,$$

$$H^i_{N+4,N+4}=-\frac{2e^{-4\beta h_2}}{h_2^2-h_1^2}\Lambda_i,H^i_{N+5,N+5}=\Xi_iA_i+A_i^{\mathrm{T}}\Xi_i-C_i^{\mathrm{T}}C_i+2\beta\Xi_i+g_i^2I,$$

$$H^i_{N+5+j,N+5+j}=-I,H^i_{i,N+5+j}=P_iA_{ij},\quad j\neq i,j=\overline{1,N},$$

$$H^i_{N+5+i,N+5+i}=-I,H^i_{i,N+5+i}=\sqrt{\xi_i}P_i,$$

$$H^i_{2N+5+j,2N+5+j}=-I,H^i_{N+3,2N+5+j}=P_iA_{ij},\quad j\neq i,j=\overline{1,N},$$

$$H^i_{2N+5+i,2N+5+i}=-I,H^i_{N+3,2N+5+i}=\sqrt{\xi_i}P_i,$$

$$H^i_{3N+5+j,3N+5+j}=-I,H^i_{N+5,3N+5+j}=\Xi_iA_{ij},\quad j\neq i,j=\overline{1,N},$$

$$H^i_{3N+5+i,3N+5+i}=-I,H^i_{N+5,3N+5+i}=\sqrt{\xi_i}\Xi_i,$$

$$\alpha_1=\min_{i=\overline{1,N}}\{\lambda_{\min}(\Xi_i)\},$$

$$\alpha_2=\max_{i=\overline{1,N}}\{\lambda_{\max}(\Xi_i)+\lambda_{\max}(P_i)+\beta^{-1}\lambda_{\max}(Q_i)+(h_1^3+h_2^3)\lambda_{\max}(R_i)+(h_2-h_1)^3\lambda_{\max}(U_i)+(h_2-h_1)h_2^2\lambda_{\max}(\Lambda_i)\}.$$

The following is the main result of the article, which gives sufficient conditions for the design of decentralized full-order Luenberger-type state observer for system (2.1). Essentially, the proof is based on the construction of a set of Lyapunov–Krasovskii functions satisfying Lyapunov stability theorem for time-delay systems [15].

Theorem 3.1

If there exist symmetric positive definite matrices $\Xi_i,P_i,Q_i,R_i,U_i,\Lambda_i,i=\overline{1,N}$, and matrix $S_i,i=\overline{1,N}$, such that the following LMIs hold:

$$\begin{pmatrix} H^i_{11} & H^i_{12} & \cdots & H^i_{1(4N+5)} & 0 & 0\\ * & H^i_{22} & \cdots & H^i_{2(4N+5)} & 0 & 0\\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot\\ * & * & \cdots & H^i_{(4N+5)(4N+5)} & 0 & 0\\ * & * & \cdots & * & -U_i & -S_i\\ * & * & \cdots & * & * & -U_i \end{pmatrix}<0,\quad i=\overline{1,N}, \tag{3.1}$$

then the error system (2.4) is $\beta-$stable with the observer gain $L_i=\Xi_i^{-1}C_i^T$. Moreover, the solution of this systems satisfies

$$||e(t)||\leq\sqrt{\frac{\alpha_2}{\alpha_1}}e^{-\beta t}||\varphi||_h,\quad\forall t\geq0.$$

Proof

Consider the following Lyapunov–Krasovskii functional:

$$V(t,x_t)=\sum_{i=1}^{N}\sum_{j=1}^{7}V_{ij}(t,x_t),$$

where

$$V_{i1}(t,x_t)=x_i(t)^{\mathrm{T}}P_ix_i(t)+e_i(t)^{\mathrm{T}}\Xi_ie_i(t),$$

$$V_{i2}(t,x_t)=\int_{t-h_1}^{t}e^{2\beta(s-t)}x_i(s)^{\mathrm{T}}Q_ix_i(s)ds,$$

$$V_{i3}(t,x_t)=\int_{t-h_2}^{t}e^{2\beta(s-t)}x_i(s)^{\mathrm{T}}Q_ix_i(s)ds,$$

$$V_{i4}(t,x_t)=h_1\int_{-h_1}^{0}\int_{t+s}^{t}e^{2\beta(\tau-t)}\dot{x}_i(\tau)^{\mathrm{T}}R_i\dot{x}_i(\tau)d\tau ds,$$

$$V_{i5}(t,x_t)=h_2\int_{-h_2}^{0}\int_{t+s}^{t}e^{2\beta(\tau-t)}\dot{x}_i(\tau)^{\mathrm{T}}R_i\dot{x}_i(\tau)d\tau ds,$$

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$$V_{i6}(t,x_t)=(h_2-h_1)\times\int_{-h_2}^{-h_1}\int_{t+s}^{t}e^{2\beta(\tau-t)}\dot{x}_i(\tau)^{\mathrm{T}}U_i\dot{x}_i(\tau)d\tau ds.$$

$$V_{i7}=\int_{-h_2}^{-h_1}\int_{\theta}^{0}\int_{t+s}^{t}e^{2\beta(\tau+s-t)}\dot{x}_i(\tau)^{\mathrm{T}}\Lambda_i\dot{x}_i(\tau)d\tau dsd\theta.$$

Taking the derivative of $V(t,x_t)$ in t along the solution of the system, we have

$$\dot{V}_{i1}(.)=2x_i(t)^{\mathrm{T}}P_i\dot{x}_i(t)+2e_i(t)^{\mathrm{T}}\Xi_i\dot{e}_i(t)$$
$$=2x_i(t)^{\mathrm{T}}P_i\left[A_ix_i(t)+\sum_{j\neq i,j=1}^{N}A_{ij}x_j(t-h_{ij}(t))+f_i(\cdot)\right]$$
$$+2e_i(t)^{\mathrm{T}}\Xi_i\Bigg[A_ie_i(t)-L_i[C_ie_i(t)+g_i(t,x_i(t))-g_i(t,\hat{x}_i(t))]$$
$$+\sum_{j\neq i,j=1}^{N}A_{ij}x_j(t-h_{ij}(t))+f_i(\cdot)\Bigg]$$
$$\dot{V}_{i2}(.)=x_i(t)^{\mathrm{T}}Q_ix_i(t)-2\beta V_{i2}(.)-e^{-2\beta h_1}x_i(t-h_1)^{\mathrm{T}}Q_ix_i(t-h_1),$$
$$\dot{V}_{i3}(.)=x_i(t)^{\mathrm{T}}Q_ix_i(t)-2\beta V_{i3}(.)-e^{-2\beta h_2}x_i(t-h_2)^{\mathrm{T}}Q_ix_i(t-h_2),$$
$$\dot{V}_{i4}(.)\leq h_1^2\dot{x}_i(t)^{\mathrm{T}}R_i\dot{x}_i(t)-2\beta V_{i4}(.)-h_1e^{-2\beta h_1}\int_{t-h_1}^{t}\dot{x}_i(s)^{\mathrm{T}}R_i\dot{x}_i(s)ds,$$
$$\dot{V}_{i5}(.)\leq h_2^2\dot{x}_i(t)^{\mathrm{T}}R_i\dot{x}_i(t)-2\beta V_{i5}(.)-h_2e^{-2\beta h_2}\int_{t-h_2}^{t}\dot{x}_i(s)^{\mathrm{T}}R_i\dot{x}_i(s)ds,$$
$$\dot{V}_{i6}(.)\leq(h_2-h_1)^2\dot{x}_i(t)^{\mathrm{T}}U_i\dot{x}_i(t)-2\beta V_{i6}(.)$$
$$-(h_2-h_1)e^{-2\beta h_2}\int_{t-h_2}^{t-h_1}\dot{x}_i(s)^{\mathrm{T}}U_i\dot{x}_i(s)ds,$$
$$\dot{V}_{i7}(.)\leq(h_2-h_1)h_2\dot{x}_i(t)^{\mathrm{T}}\Lambda_i\dot{x}_i(t)-2\beta V_{i7}(.)$$
$$-e^{-4\beta h_2}\int_{-h_2}^{-h_1}\int_{t+\theta}^{t}\dot{x}_i(s)^{\mathrm{T}}\Lambda_i\dot{x}_i(s)dsd\theta.$$

We first estimate $\dot{V}_{i1}(.)$ as follows. Using Cauchy matrix inequality (Proposition 2.4) gives

$$2x_i(t)^{\mathrm{T}}P_i\left[\sum_{j\neq i,j=1}^{N}A_{ij}x_j(t-h_{ij}(t))\right]\leq\sum_{j\neq i,j=1}^{N}x_i(t)^{\mathrm{T}}P_iA_{ij}A_{ij}^{\mathrm{T}}P_ix_i(t)$$
$$+\sum_{j\neq i,j=1}^{N}x_j(t-h_{ij}(t))^{\mathrm{T}}x_j(t-h_{ij}(t)),$$
$$2e_i(t)^{\mathrm{T}}\Xi_i\left[\sum_{j\neq i,j=1}^{N}A_{ij}x_j(t-h_{ij}(t))\right]\leq\sum_{j\neq i,j=1}^{N}e_i(t)^{\mathrm{T}}\Xi_iA_{ij}A_{ij}^{\mathrm{T}}\Xi_ie_i(t)$$
$$+\sum_{j\neq i,j=1}^{N}x_j(t-h_{ij}(t))^{\mathrm{T}}x_j(t-h_{ij}(t)).$$

Then from the condition (2.2), it follows

$$2x_i(t)^{\mathrm{T}}P_if_i(\cdot)\leq2||x_i(t)^{\mathrm{T}}P_i||\left[a_i||x_i(t)||+\sum_{j\neq i,j=1}^{N}a_{ij}||x_j(t-h_{ij}(t))||\right]$$
$$\leq\xi_i||x_i(t)^{\mathrm{T}}P_i||^2+a_i||x_i(t)||^2+\sum_{j\neq i,j=1}^{N}a_{ij}||x_j(t-h_{ij}(t))||^2,\tag{3.2}$$
$$2\dot{e}_i(t)^{\mathrm{T}}\Xi_if_i(\cdot)\leq\xi_i||e_i(t)^{\mathrm{T}}\Xi_i||^2+a_i||x_i(t)||^2+\sum_{j\neq i,j=1}^{N}a_{ij}||x_j(t-h_{ij}(t))||^2.\tag{3.3}$$

Taking $L_i=\Xi_i^{-1}C_i^{\mathrm{T}}$ and using Cauchy matrix inequality (Proposition 2.4) and the condition (2.2) again leads to

$$\begin{aligned}&-2e_i(t)^{\mathrm{T}}\Xi_iL_i(g_i(t,x_i(t)))-g_i(t,\hat{x}_i(t)))\\&=-2e_i(t)^{\mathrm{T}}C_i^{\mathrm{T}}(g_i(t,x_i(t)))-g_i(t,\hat{x}_i(t)))\\&\leq e_i(t)^{\mathrm{T}}C_i^{\mathrm{T}}C_ie_i(t)+||g_i(t,x_i(t)))-g_i(t,\hat{x}_i(t))||^2\\&\leq e_i(t)^{\mathrm{T}}C_i^{\mathrm{T}}C_ie_i(t)+g_i^2e_i(t)^{\mathrm{T}}e_i(t).\end{aligned}\tag{3.4}$$

From (3.2)–(3.4), we have

$$\dot{V}_{i1}(.)\leq x_i(t)^{\mathrm{T}}[P_iA_i+A_i^{\mathrm{T}}P_i]x_i(t)$$
$$+\sum_{j\neq i,j=1}^{N}x_i(t)^{\mathrm{T}}P_iA_{ij}A_{ij}^{\mathrm{T}}P_ix_i(t)+\sum_{j\neq i,j=1}^{N}x_j(t-h_{ij}(t))^{\mathrm{T}}x_j(t-h_{ij}(t))$$
$$+\xi_i||x_i(t)^{\mathrm{T}}P_i||^2+a_i||x_i(t)||^2+\sum_{j\neq i,j=1}^{N}a_{ij}||x_j(t-h_{ij}(t))||^2$$
$$+e_i(t)^{\mathrm{T}}[\Xi_iA_i+A_i^{\mathrm{T}}\Xi_i-2C_i^{\mathrm{T}}C_i]e_i(t)+e_i(t)^{\mathrm{T}}C_i^{\mathrm{T}}C_ie_i(t)+g_i^2e_i(t)^{\mathrm{T}}e_i(t)$$
$$+\sum_{j\neq i,j=1}^{N}e_i(t)^{\mathrm{T}}\Xi_iA_{ij}A_{ij}^{\mathrm{T}}\Xi_ie_i(t)+\sum_{j\neq i,j=1}^{N}x_j(t-h_{ij}(t))^{\mathrm{T}}x_j(t-h_{ij}(t))$$
$$+\xi_i||e_i(t)^{\mathrm{T}}\Xi_i||^2+a_i||x_i(t)||^2+\sum_{j\neq i,j=1}^{N}a_{ij}||x_j(t-h_{ij}(t))||^2.\tag{3.5}$$

To estimate $\dot{V}_{i4}(.),\dot{V}_{i5}(.)$, we apply Proposition 2.2 and the Newton–Leibniz formula, for $k=1,2$, to obtain

$$-h_k\int_{t-h_k}^{t}\dot{x}_i(s)^{\mathrm{T}}R_i\dot{x}_i(s)ds\leq-\left[\int_{t-h_k}^{t}\dot{x}_i(s)ds\right]^{\mathrm{T}}R_i\left[\int_{t-h_k}^{t}\dot{x}_i(s)ds\right]$$
$$=-[x_i(t)-x_i(t-h_k)]^{\mathrm{T}}R_i[x_i(t)-x_i(t-h_k)].$$

Therefore, we have

$$\dot{V}_{i4}(.)\leq h_1^2\dot{x}_i(t)^{\mathrm{T}}R_i\dot{x}_i(t)-2\beta V_{i4}-e^{-2\beta h_1}[x_i(t)-x_i(t-h_1)]^{\mathrm{T}}R_i[x_i(t)-x_i(t-h_1)],\tag{3.6}$$
$$\dot{V}_{i5}(.)\leq h_2^2\dot{x}_i(t)^{\mathrm{T}}R_i\dot{x}_i(t)-2\beta V_{i5}-e^{-2\beta h_2}[x_i(t)-x_i(t-h_2)]^{\mathrm{T}}R_i[x_i(t)-x_i(t-h_2)].\tag{3.7}$$

We now estimate $\dot{V}_{i6}(.)$ as follows. The integral $\int_{t-h_2}^{t-h_1}\dot{x}_i{}^{\mathrm{T}}(s)U_i\dot{x}_i(s)ds$ decomposed as

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$$\int_{t-h_2}^{t-h_1} \dot{x}_i^{\mathrm{T}}(s)U_i\dot{x}_i(s), ds = \int_{t-h_2}^{t-h_{ji}(t)} \dot{x}_i^{\mathrm{T}}(s)U_i\dot{x}_i(s)ds + \int_{t-h_{ji}(t)}^{t-h_1} \dot{x}_i^{\mathrm{T}}(s)U_i\dot{x}_i(s)ds,$$

and using Proposition 2.2 gives

$$\begin{aligned}
&-(h_2-h_1)\int_{t-h_2}^{t-h_{ji}(t)} \dot{x}_i^{\mathrm{T}}(s)U_i\dot{x}_i(s)ds \\
&\le -\frac{h_2-h_1}{h_2-h_{ji}(t)}[x_i(t-h_{ji}(t))-x_i(t-h_2)]^{\mathrm{T}}U_i[x_i(t-h_{ji}(t))-x_i(t-h_2)], \\
&-(h_2-h_1)\int_{t-h_{ji}(t)}^{t-h_1} \dot{x}_i^{\mathrm{T}}(s)U_i\dot{x}_i(s)ds \\
&\le -\frac{h_2-h_1}{h_{ji}(t)-h_1}[x_i(t-h_1)-x_i(t-h_{ji}(t))]^{\mathrm{T}}U_i[x_i(t-h_1)-x_i(t-h_{ji}(t))].
\end{aligned}$$

Then, we have

$$\begin{aligned}
&-(h_2-h_1)\int_{t-h_2}^{t-h_1} \dot{x}_i^{\mathrm{T}}(s)U_i\dot{x}_i(s)ds \\
&\le -\frac{h_2-h_1}{h_2-h_{ji}(t)}[x_i(t-h_{ji}(t))-x_i(t-h_2)]^{\mathrm{T}}U_i[x_i(t-h_{ji}(t))-x_i(t-h_2)] \\
&-\frac{h_2-h_1}{h_{ji}(t)-h_1}[x_i(t-h_1)-x_i(t-h_{ji}(t))]^{\mathrm{T}}U_i[x_i(t-h_1)-x_i(t-h_{ji}(t))].
\end{aligned} \tag{3.8}$$

Let

$$r_1=\frac{h_2-h_{ji}(t)}{h_2-h_1}, \quad r_2=\frac{h_{ji}(t)-h_1}{h_2-h_1},$$

and

$$\begin{aligned}
f_1(t)&=[x_i(t-h_{ji}(t))-x_i(t-h_2)]^{\mathrm{T}}U_i[x_i(t-h_{ji}(t))-x_i(t-h_2)], \\
f_2(t)&=[x_i(t-h_1)-x_i(t-h_{ji}(t))]^{\mathrm{T}}U_i[x_i(t-h_1)-x_i(t-h_{ji}(t))], \\
g_{1,2}(t)&=[x_i(t-h_{ji}(t))-x_i(t-h_2)]^{\mathrm{T}}S_i[x_i(t-h_1)-x_i(t-h_{ji}(t))], \\
g_{2,1}(t)&=[x_i(t-h_1)-x_i(t-h_{ji}(t))]^{\mathrm{T}}S_i^{\mathrm{T}}[x_i(t-h_{ji}(t))-x_i(t-h_2)].
\end{aligned}$$

It follows from condition (3.1) that $\begin{bmatrix} U_i & S_i \\ S_i^{\mathrm{T}} & U_i \end{bmatrix} > 0$, and hence

$$\begin{aligned}
&\begin{bmatrix} f_1(t) & g_{1,2}(t) \\ g_{1,2}(t) & f_2(t) \end{bmatrix} \\
&= \begin{bmatrix} [x_i(t-h_{ji}(t))-x_i(t-h_2)]^{\mathrm{T}} & 0 \\ 0 & [x_i(t-h_1)-x_i(t-h_{ji}(t))]^{\mathrm{T}} \end{bmatrix} \\
&\times \begin{bmatrix} U_i & S_i \\ S_i^{\mathrm{T}} & U_i \end{bmatrix} \\
&\times \begin{bmatrix} [x_i(t-h_{ji}(t))-x_i(t-h_2)] & 0 \\ 0 & [x_i(t-h_1)-x_i(t-h_{ji}(t))] \end{bmatrix} \ge 0.
\end{aligned}$$

Moreover, note that $g_{1,2}(t)=g_{2,1}(t)$ and $r_1+r_2=1, r_1>0$, $r_2>0$. Using Proposition 2.3 and the inequality (3.8) gives

$$\begin{aligned}
&-(h_2-h_1)\int_{t-h_2}^{t-h_1} \dot{x}_i^{\mathrm{T}}(s)U_i\dot{x}_i(s)ds \le -\frac{1}{r_1}f_1(t)-\frac{1}{r_2}f_2(t) \\
&\le -f_1(t)-f_2(t)-g_{1,2}(t)-g_{2,1}(t) \\
&=-[x_i(t-h_{ji}(t))-x_i(t-h_2)]^{\mathrm{T}}U_i[x_i(t-h_{ji}(t))-x_i(t-h_2)] \\
&-[x_i(t-h_1)-x_i(t-h_{ji}(t))]^{\mathrm{T}}U_i[x_i(t-h_1)-x_i(t-h_{ji}(t))] \\
&-[x_i(t-h_{ji}(t))-x_i(t-h_2)]^{\mathrm{T}}S_i[x_i(t-h_1)-x_i(t-h_{ji}(t))] \\
&-[x_i(t-h_1)-x_i(t-h_{ji}(t))]^{\mathrm{T}}S_i^{\mathrm{T}}[x_i(t-h_{ji}(t))-x_i(t-h_2)].
\end{aligned} \tag{3.9}$$

Taking $j=1,2,..,N, j \ne i$ the inequality (3.9) implies

$$\begin{aligned}
&-(N-1)(h_2-h_1)\int_{t-h_2}^{t-h_1} \dot{x}_i^{\mathrm{T}}(s)U_i\dot{x}_i(s)ds \\
&\le -\sum_{j=1,j\ne i}^{N}[x_i(t-h_{ji}(t))-x_i(t-h_2)]^{\mathrm{T}}U_i[x_i(t-h_{ji}(t))-x_i(t-h_2)] \\
&-\sum_{j=1,j\ne i}^{N}[x_i(t-h_1)-x_i(t-h_{ji}(t))]^{\mathrm{T}}U_i[x_i(t-h_1)-x_i(t-h_{ji}(t))] \\
&-\sum_{j=1,j\ne i}^{N}[x_i(t-h_{ji}(t))-x_i(t-h_2)]^{\mathrm{T}}S_i[x_i(t-h_1)-x_i(t-h_{ji}(t))] \\
&-\sum_{j=1,j\ne i}^{N}[x_i(t-h_1)-x_i(t-h_{ji}(t))]^{\mathrm{T}}S_i^{\mathrm{T}}[x_i(t-h_{ji}(t))-x_i(t-h_2)].
\end{aligned}$$

Note that when $h_{ji}(t)=h_1$ or $h_{ji}(t)=h_2$, we have

$$[x_i(t-h_1)-x_i(t-h_{ji}(t))]^{\mathrm{T}}=0 \quad \text{or} \quad [x_i(t-h_{ji}(t))-x_i(t-h_2)]=0,$$

respectively, so the relation (3.10) still holds. Thus, we obtain the estimation of $\dot{V}_{i6}(.)$ as

$$\begin{aligned}
&\dot{V}_{i6}(.) \le (h_2-h_1)^2\dot{x}_i(t)^{\mathrm{T}}U_i\dot{x}_i(t)-2\beta V_{i6} \\
&-\frac{e^{-2\beta h_2}}{N-1}\sum_{j=1,j\ne i}^{N}[x_i(t-h_{ji}(t))-x_i(t-h_2)]^{\mathrm{T}}U_i[x_i(t-h_{ji}(t))-x_i(t-h_2)] \\
&-\frac{e^{-2\beta h_2}}{N-1}\sum_{j=1,j\ne i}^{N}[x_i(t-h_1)-x_i(t-h_{ji}(t))]^{\mathrm{T}}U_i[x_i(t-h_1)-x_i(t-h_{ji}(t))] \\
&-\frac{e^{-2\beta h_2}}{N-1}\sum_{j=1,j\ne i}^{N}[x_i(t-h_{ji}(t))-x_i(t-h_2)]^{\mathrm{T}}S_i[x_i(t-h_1)-x_i(t-h_{ji}(t))] \\
&-\frac{e^{-2\beta h_2}}{N-1}\sum_{j=1,j\ne i}^{N}[x_i(t-h_1)-x_i(t-h_{ji}(t))]^{\mathrm{T}}S_i^{\mathrm{T}}[x_i(t-h_{ji}(t))-x_i(t-h_2)].
\end{aligned} \tag{3.10}$$

To estimate $\dot{V}_{i7}(.)$, we apply Proposition 2.2 for the estimation of the double integral

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$$-e^{-4\beta h_2}\int_{-h_2}^{-h_1}\int_{t+\theta}^{t}\dot{x}_i(s)^{\mathrm{T}}\Lambda_i\dot{x}_i(s)dsd\theta$$
$$\leq -e^{-4\beta h_2}\frac{2}{h_2^2-h_1^2}\left(\int_{-h_2}^{-h_1}\int_{t+\theta}^{t}\dot{x}_i(s)dsd\theta\right)^{\mathrm{T}}\Lambda_i\left(\int_{-h_2}^{-h_1}\int_{t+\theta}^{t}\dot{x}_i(s)dsd\theta\right)$$
$$\leq -\frac{2e^{-4\beta h_2}}{h_2^2-h_1^2}\left((h_2-h_1)x_i(t)-\int_{t-h_2}^{t-h_1}x_i(\theta)d\theta\right)^{\mathrm{T}}$$
$$\times\Lambda_i\left((h_2-h_1)x_i(t)-\int_{t-h_2}^{t-h_1}x_i(\theta)d\theta\right),$$

and hence

$$\begin{aligned}\dot{V}_{i7}(.)\leq\ &(h_2-h_1)h_2\dot{x}_i(t)^{\mathrm{T}}\Lambda_i\dot{x}_i(t)-2\beta V_{i7}\\&-\frac{2e^{-4\beta h_2}}{h_2^2-h_1^2}\left((h_2-h_1)x_i(t)-\int_{t-h_2}^{t-h_1}x_i(\theta)d\theta\right)^{\mathrm{T}}\\&\times\Lambda_i\left((h_2-h_1)x_i(t)-\int_{t-h_2}^{t-h_1}x_i(\theta)d\theta\right).\end{aligned}\tag{3.11}$$

Finally, we derive the estimation from $\dot{V}(t,x_t)$ by (3.5)–(3.7), (3.10), (3.11) as

$$\begin{aligned}&\dot{V}(t,x_t)+2\beta V(t,x_t)\leq x_i(t)^{\mathrm{T}}[P_iA_i+A_i^{\mathrm{T}}P_i+2\beta P_i]x_i(t)\\&+\sum_{j\neq i,j=1}^{N}x_i(t)^{\mathrm{T}}P_iA_{ij}A_{ij}^{\mathrm{T}}P_ix_i(t)+\sum_{j\neq i,j=1}^{N}x_j(t-h_{ij}(t))^{\mathrm{T}}x_j(t-h_{ij}(t))\\&+\xi_i||x_i(t)^{\mathrm{T}}P_i||^2+a_i||x_i(t)||^2+\sum_{j\neq i,j=1}^{N}a_{ij}||x_j(t-h_{ij}(t))||^2\\&+e_i(t)^{\mathrm{T}}[\Xi_iA_i+A_i^{\mathrm{T}}\Xi_i-2C_i^{\mathrm{T}}C_i]e_i(t)+e_i(t)^{\mathrm{T}}C_i^{\mathrm{T}}C_ie_i(t)+g_i^2e_i(t)^{\mathrm{T}}e_i(t)\\&\quad+\sum_{j\neq i,j=1}^{N}e_i(t)^{\mathrm{T}}\Xi_iA_{ij}A_{ij}^{\mathrm{T}}\Xi_ie_i(t)+\sum_{j\neq i,j=1}^{N}x_j(t-h_{ij}(t))^{\mathrm{T}}x_j(t-h_{ij}(t))\\&\quad+\xi_i||e_i(t)^{\mathrm{T}}\Xi_i||^2+a_i||x_i(t)||^2+\sum_{j\neq i,j=1}^{N}a_{ij}||x_j(t-h_{ij}(t))||^2\\&\quad+x_i(t)^{\mathrm{T}}Q_ix_i(t)-e^{-2\beta h_1}x_i(t-h_1)^{\mathrm{T}}Q_ix_i(t-h_1)\\&\quad+x_i(t)^{\mathrm{T}}Q_ix_i(t)-e^{-2\beta h_2}x_i(t-h_2)^{\mathrm{T}}Q_ix_i(t-h_2)\\&\quad+h_1^2\dot{x}_i(t)^{\mathrm{T}}R_i\dot{x}_i(t)-e^{-2\beta h_1}[x_i(t)-x_i(t-h_1)]^{\mathrm{T}}R_i[x_i(t)-x_i(t-h_1)]\\&+h_2^2\dot{x}_i(t)^{\mathrm{T}}R_i\dot{x}_i(t)-e^{-2\beta h_2}[x_i(t)-x_i(t-h_2)]^{\mathrm{T}}R_i[x_i(t)-x_i(t-h_2)]\\&+(h_2-h_1)^2\dot{x}_i(t)^{\mathrm{T}}U_i\dot{x}_i(t)\\&-\frac{e^{-2\beta h_2}}{N-1}\sum_{j=1,j\neq i}^{N}\left[x_i(t-h_{ji}(t))-x_i(t-h_2)\right]^{\mathrm{T}}U_i\left[x_i(t-h_{ji}(t))-x_i(t-h_2)\right]\\&-\frac{e^{-2\beta h_2}}{N-1}\sum_{j=1,j\neq i}^{N}\left[x_i(t-h_1)-x_i(t-h_{ji}(t))\right]^{\mathrm{T}}U_i\left[x_i(t-h_1)-x_i(t-h_{ji}(t))\right]\\&-\frac{e^{-2\beta h_2}}{N-1}\sum_{j=1,j\neq i}^{N}\left[x_i(t-h_{ji}(t))-x_i(t-h_2)\right]^{\mathrm{T}}S_i\left[x_i(t-h_1)-x_i(t-h_{ji}(t))\right]\\&-\frac{e^{-2\beta h_2}}{N-1}\sum_{j=1,j\neq i}^{N}\left[x_i(t-h_1)-x_i(t-h_{ji}(t))\right]^{\mathrm{T}}S_i^T\\&\left[x_i(t-h_{ji}(t))-x_i(t-h_2)\right]\\&+(h_2-h_1)h_2\dot{x}_i(t)^T\Lambda_i\dot{x}_i(t)\\&-\frac{2e^{-4\beta h_2}}{h_2^2-h_1^2}\left((h_2-h_1)x_i(t)-\int_{t-h_2}^{t-h_1}x_i(\theta)d\theta\right)^T\\&\Lambda_i\left((h_2-h_1)x_i(t)-\int_{t-h_2}^{t-h_1}x_i(\theta)d\theta\right).\end{aligned}\tag{3.12}$$

From the Eq. (2.1), we have

$$\begin{aligned}&0=-2\dot{x}_i(t)^{\mathrm{T}}P_i[\dot{x}_i(t)-A_ix_i(t)-\sum_{j\neq i,j=1}^{N}A_{ij}x_j(t-h_{ij}(t))-f_i(\cdot)]\\&\leq-2\dot{x}_i(t)^{\mathrm{T}}P_i[\dot{x}_i(t)-A_ix_i(t)]\\&+\xi_i||\dot{x}_i(t)^{\mathrm{T}}P_i||^2+a_i||x_i(t)||^2+\sum_{j\neq i,j=1}^{N}a_{ij}||x_j(t-h_{ij}(t))||^2\\&+\sum_{j\neq i,j=1}^{N}\dot{x}_i(t)^{\mathrm{T}}P_iA_{ij}A_{ij}^{\mathrm{T}}P_i\dot{x}_i(t)\\&+\sum_{j\neq i,j=1}^{N}x_j(t-h_{ij}(t))^{\mathrm{T}}x_j(t-h_{ij}(t)),\end{aligned}\tag{3.13}$$

because of

$$2\dot{x}_i(t)^{\mathrm{T}}P_i\left[\sum_{j\neq i,j=1}^{N}A_{ij}x_j(t-h_{ij}(t))\right]\leq\sum_{j\neq i,j=1}^{N}\dot{x}_i(t)^{\mathrm{T}}P_iA_{ij}A_{ij}^{\mathrm{T}}P_i\dot{x}_i(t)$$
$$+\sum_{j\neq i,j=1}^{N}x_j(t-h_{ij}(t))^{\mathrm{T}}x_j(t-h_{ij}(t)),$$
$$2\dot{x}_i(t)^{\mathrm{T}}P_if_i(\cdot)\leq\xi_i||\dot{x}_i(t)^{\mathrm{T}}P_i||^2+a_i||x_i(t)||^2+\sum_{j\neq i,j=1}^{N}a_{ij}||x_j(t-h_{ij}(t))||^2.$$

Adding the inequality (3.13) into the left side of (3.12) with the resulting equalities:

$$\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N}x_j(t-h_{ij}(t))^{\mathrm{T}}x_j(t-h_{ij}(t))=\sum_{j=1}^{N}\sum_{i=1,i\neq j}^{N}x_i(t-h_{ji}(t))^{\mathrm{T}}x_i(t-h_{ji}(t))$$
$$=\sum_{i=1}^{N}\left[\sum_{j=1,i\neq j}^{N}x_i(t-h_{ji}(t))^{\mathrm{T}}x_i(t-h_{ji}(t))\right],$$
$$\sum_{i=1}^{N}\sum_{j\neq i,j=1}^{N}a_{ij}||x_j(t-h_{ij}(t))||^2=\sum_{i=1}^{N}\sum_{j\neq i,j=1}^{N}a_{ji}||x_i(t-h_{ji}(t))||^2,$$

we obtain

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$$\dot{V}(t,x_t)+2\beta V(t,x_t) \le \sum_{i=1}^{N} \xi_i(t)^{\mathrm{T}} M^i \xi_i(t), \tag{3.14}$$

where for $v_i^j = x_i(t-h_{ji}(t))^{\mathrm{T}}, i \neq j, v_i^i = x_i(t)^{\mathrm{T}}$,

$$\xi_i(t)^{\mathrm{T}} = \left[v_i^1,\ldots,v_i^N x_i(t-h_1)^{\mathrm{T}} x_i(t-h_2)^{\mathrm{T}} \dot{x}_i(t)^{\mathrm{T}} \int_{t-h_2}^{t-h_1} x_i(\theta)^{\mathrm{T}} d\theta, e_i(t)^{\mathrm{T}} \right],$$

$$M^i = \begin{bmatrix} M_{11}^i & M_{12}^i & . & . & . & M_{1(N+5)}^i \\ * & M_{22}^i & . & . & . & M_{2(N+5)}^i \\ . & . & . & . & . & . \\ * & * & . & . & . & M_{(N+5)(N+5)}^i \end{bmatrix}, \quad i=\overline{1,N},$$

$$M_{i,i}^i = P_i A_i + A_i^{\mathrm{T}} P_i + 2\beta P_i + 2Q_i - e^{-2\beta h_1} R_i - e^{-2\beta h_2} R_i$$
$$-\frac{2e^{-4\beta h_2}(h_2-h_1)}{h_2+h_1}\Lambda_i + 3a_i I + \sum_{j\neq i,j=1}^{N} P_i A_{ij} A_{ij}^{\mathrm{T}} P_i + \xi_i P_i^2,$$
$$M_{i,j}^i = 0, j=\overline{1,N}, j\neq i, M_{i,N+1}^i = e^{-2\beta h_1} R_i, M_{i,N+2}^i = e^{-2\beta h_2} R_i,$$
$$M_{i,N+3}^i = A_i^{\mathrm{T}} P_i, M_{i,N+4}^i = \frac{2e^{-4\beta h_2}}{h_2+h_1}\Lambda_i, M_{i,N+5}^i = 0,$$
$$M_{j,j}^i = (3a_{ji}+3)I - \frac{2e^{-2\beta h_2}}{N-1} U_i + \frac{e^{-2\beta h_2}}{N-1}(S_i + S_i^{\mathrm{T}}),$$
$$j\neq i, M_{j,k}^i = 0, j\neq k, k=\overline{1,N}$$
$$M_{j,(N+1)}^i = \frac{e^{-2\beta h_2}}{N-1} U_i - \frac{e^{-2\beta h_2}}{N-1} S_i, M_{j,(N+2)}^i = \frac{e^{-2\beta h_2}}{N-1} U_i - \frac{e^{-2\beta h_2}}{N-1} S_i^T,$$
$$M_{j,(N+3)}^i = 0, M_{j,(N+4)}^i = 0, M_{j,(N+5)}^i = 0, j\neq i, j=\overline{1,N},$$
$$M_{N+1,N+1}^i = -e^{-2\beta h_1} Q_i - e^{-2\beta h_1} R_i - e^{-2\beta h_2} U_i,$$
$$M_{N+1,N+2}^i = e^{-2\beta h_2} S_i^{\mathrm{T}}, M_{N+1,N+3}^i = 0, M_{N+1,N+4}^i = 0, M_{N+1,N+5}^i = 0,$$
$$M_{N+2,N+2}^i = -e^{-2\beta h_2} Q_i - e^{-2\beta h_2} R_i - e^{-2\beta h_2} U_i,$$
$$M_{N+2,N+3}^i = M_{N+2,N+4}^i = M_{N+2,N+5}^i = 0,$$
$$M_{N+3,N+3}^i = (h_1^2 + h_2^2) R_i + (h_2-h_1)^2 U_i + (h_2-h_1) h_2 \Lambda_i$$
$$-2P_i + \sum_{j\neq i,j=1}^{N} P_i A_{ij} A_{ij}^T P_i + \xi_i P_i^2,$$
$$M_{N+3,N+4}^i = 0, M_{N+3,N+5}^i = 0,$$
$$M_{N+4,N+4}^i = -\frac{2e^{-4\beta h_2}}{h_2^2 - h_1^2}\Lambda_i, M_{N+4,N+5}^i = 0,$$
$$M_{N+5,N+5}^i = \Xi_i A_i + A_i^{\mathrm{T}} \Xi_i - C_i^{\mathrm{T}} C_i + 2\beta \Xi_i + g_i^2 I$$
$$+ \sum_{j\neq i,j=1}^{N} \Xi_i A_{ij} A_{ij}^{\mathrm{T}} \Xi_i + \xi_i \Xi_i^2.$$

Using the Schur complement lemma, Proposition 2.1, the condition (3.1) leads to $M^i < 0, \forall i=\overline{1,N}$ and from the inequality (3.14), it follows that

$$\dot{V}(t,x_t) + 2\beta V(t,x_t) \le 0, \quad \forall t \ge 0,$$

which gives

$$V(t,x_t) \le V(0,x_0) e^{-2\beta t}, \quad \forall t \ge 0.$$

It is easy to verify that

$$\alpha_1 \sum_{i=1}^{N} ||e_i(t)||^2 \le V(t,x_t), \quad V(0,x_0) \le \alpha_2 \sum_{i=1}^{N} ||\varphi_i||_{C_1}^2. \tag{3.15}$$

Taking inequalities (3.14), (3.15) in account, we finally obtain that

$$\alpha_1 \sum_{i=1}^{N} ||e_i(t)||^2 \le V(t,x_t) \le \alpha_2 \sum_{i=1}^{N} ||\varphi_i||_{C_1}^2 e^{-2\beta t}, \quad \text{for all } t \ge 0,$$

and hence

$$||e(t)|| \le \sqrt{\frac{\alpha_2}{\alpha_1}} e^{-\beta t} ||\varphi||_h, \quad \forall t \ge 0,$$

which implies that the error solution of the closed-loop system is $\beta-$ stable. ■

Remark 3.1.

Theorem 3.1 provides sufficient conditions for designing state observer of the nonlinear large-scale system (2.1) in terms of the solutions of LMIs, which guarantees the error system to be exponentially stable with a prescribed decay rate β. Note that the time-varying delays are nondifferentiable, therefore, the methods proposed in [10,11] are not applicable to system (2.1). The LMI condition (3.1) depends on parameters of the system under consideration as well as the delay bounds. The feasibility of the LMIs can be tested by the reliable and efficient Matlab LMI Control Toolbox [19].

4. AN ILLUSTRATIVE EXAMPLE

In the following, we give a numerical example to show the validity of the design of decentralized state observer presented in this article.

Example 1.

Consider a large-scale model (2.1) composed of three machine subsystems [1] as follows:

$$\begin{cases} \dot{x}_1(t) = A_1 x_1(t) + A_{12} x_2(t-h_{12}(t)) + A_{13} x_3(t-h_{13}(t)) \\ \quad + f_1(t, x_1(t), x_2(t-h_{12}(t)), x_3(t-h_{13}(t))), \\ z_1(t) = C_1 x_1(t) + g_1(t, x_1(t)), t \ge 0, \\ x_1(\theta) = \varphi_1(\theta), \theta \in [-h_2, 0], \end{cases}$$

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$$\begin{cases} \dot{x}_2(t)=A_2x_2(t)+A_{21}x_1(t-h_{21}(t))+A_{23}x_3(t-h_{23}(t)) \\ \quad +f_2(t,x_2(t),x_1(t-h_{21}(t)),x_3(t-h_{23}(t))), \\ z_2(t)=C_2x_2(t)+g_2(t,x_2(t)),\ t\geq 0, \\ x_2(\theta)=\varphi_2(\theta),\ \theta\in[-h_2,0], \end{cases}$$

$$\begin{cases} \dot{x}_3(t)=A_3x_3(t)+A_{31}x_1(t-h_{31}(t))+A_{32}x_2(t-h_{32}(t)) \\ \quad +f_3(t,x_3(t),x_1(t-h_{31}(t)),x_2(t-h_{32}(t))), \\ z_3(t)=C_3x_3(t)+g_3(t,x_3(t)),\ t\geq 0, \\ x_3(\theta)=\varphi_3(\theta),\ \theta\in[-h_2,0], \end{cases}$$

where the absolute rotor angle and angular velocity of the machine in each subsystem are denoted by $x_i=(x_{i1},x_{i2})^{\mathrm{T}}, i=1, 2,3$; the ith system coefficient A_i; the control and nonlinear perturbations $f_i(\cdot)$; and the modulus of the transfer admittance A_{ij}; the initial input φ_i; the time-varying delays $h_{ij}(t)$ between each two machines in the subsystem:

$$h_{12}=\begin{cases} 1+\sin t, & t\in H, \\ 1, & t\notin H, \end{cases} \qquad h_{13}=\begin{cases} 2+0.5\sin t, & t\in H, \\ 2, & t\notin H, \end{cases}$$

$$h_{21}=\begin{cases} 1.5+1\sin t, & t\in H, \\ 1.5, & t\notin H, \end{cases} \qquad h_{23}=\begin{cases} 1+\sin t, & t\in H, \\ 1, & t\notin H, \end{cases}$$

$$h_{31}=\begin{cases} 1.8+0.5\sin t, & t\in H, \\ 1.8, & t\notin H, \end{cases} \qquad h_{32}=\begin{cases} 2.4+0.1\sin t, & t\in H, \\ 2.4, & t\notin H, \end{cases}$$

$$H=\cup_{k\in N}(2k\pi,(2k+1)\pi),$$

$$A_1=\begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix}, A_{12}=\begin{bmatrix} 0.01 & 0.01 \\ 0.01 & 0.01 \end{bmatrix}, A_{13}=\begin{bmatrix} 0.01 & -0.01 \\ -0.01 & 0.01 \end{bmatrix},$$

$$C_1=[2\quad 1],$$

$$A_2=\begin{bmatrix} -1.5 & 1 \\ 1.5 & -2 \end{bmatrix}, A_{21}=\begin{bmatrix} 0.02 & 0.01 \\ 0.03 & 0.01 \end{bmatrix}, A_{23}=\begin{bmatrix} 0.01 & 0.02 \\ 0.03 & 0.01 \end{bmatrix},$$

$$C_2=[1\quad 2],$$

$$A_3=\begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}, A_{31}=\begin{bmatrix} 0.03 & 0.01 \\ 0.03 & 0.01 \end{bmatrix}, A_{32}=\begin{bmatrix} 0.03 & 0.02 \\ 0.01 & 0.01 \end{bmatrix},$$

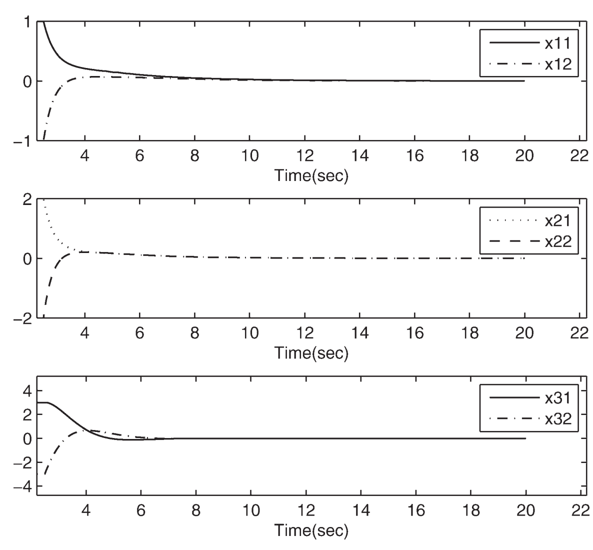
$$C_3=[2\quad 3],$$

$$f_1(.)=0.001\begin{bmatrix} \sqrt{x_{11}(t)^2+x_{21}(t-h_{12}(t))^2+x_{31}(t-h_{13}(t))^2} \\ \sqrt{x_{12}(t)^2+x_{22}(t-h_{12}(t))^2+x_{32}(t-h_{13}(t))^2} \end{bmatrix},$$

$$f_2(.)=0.001\begin{bmatrix} \sqrt{x_{21}(t)^2+x_{11}(t-h_{21}(t))^2+x_{31}(t-h_{23}(t))^2} \\ \sqrt{x_{22}(t)^2+x_{12}(t-h_{21}(t))^2+x_{32}(t-h_{23}(t))^2} \end{bmatrix},$$

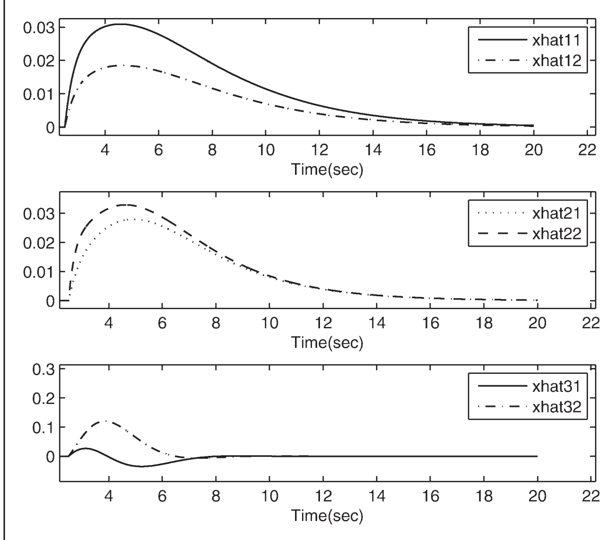
$$f_3(.)=0.001\begin{bmatrix} \sqrt{x_{31}(t)^2+x_{11}(t-h_{31}(t))^2+x_{21}(t-h_{32}(t))^2} \\ \sqrt{x_{32}(t)^2+x_{12}(t-h_{31}(t))^2+x_{22}(t-h_{32}(t))^2} \end{bmatrix},$$

FIGURE 1



Solution response of $x_1(t),x_2(t),x_3(t)$.

FIGURE 2



Solution response of $\hat{x}_1(t),\hat{x}_2(t)$, and $\hat{x}_3(t)$.

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$$g_1(.)=2\sqrt{x_{11}(t)^2+x_{12}(t)^2}, g_2(.)=2\sqrt{x_{21}(t)^2+x_{22}(t)^2}, g_3(.)$$
$$=2\sqrt{x_{31}(t)^2+x_{32}(t)^2},$$
$$a_i=a_{ij}=0.001, \forall i,j=1,2,3, i\neq j, g_1=g_2=g_3=2.$$

It is worth nothing that, the delay functions $h_{ij}(t)$ are nondifferentiable, therefore, the design method in [10,11] are not applicable to this system. Using LMI Toolbox in Matlab, the LMI (3.1) is feasible with $h_1=1$, $h_2=2.5, \beta=0.1$, and

$$E_1=\begin{bmatrix}104.8948 & -7.0715\\ -7.0715 & 112.6934\end{bmatrix}, P_1=\begin{bmatrix}75.0815 & 1.5678\\ 1.5678 & 73.5169\end{bmatrix}, Q_1=\begin{bmatrix}6.8126 & -2.4370\\ -2.4370 & 9.2501\end{bmatrix},$$

$$R_1=\begin{bmatrix}1.0850 & 0.5640\\ 0.5640 & 0.5212\end{bmatrix}, U_1=\begin{bmatrix}14.4076 & 5.5859\\ 5.5859 & 8.8227\end{bmatrix}, \Lambda_1=\begin{bmatrix}2.1217 & 0.9636\\ 0.9636 & 1.1585\end{bmatrix},$$

$$E_2=\begin{bmatrix}102.7269 & -6.6339\\ -6.6339 & 76.1245\end{bmatrix}, P_2=\begin{bmatrix}60.3863 & -1.6922\\ -1.6922 & 42.6847\end{bmatrix}, Q_2=\begin{bmatrix}2.9314 & -0.3157\\ -0.3157 & 3.0649\end{bmatrix},$$

$$R_2=\begin{bmatrix}0.9034 & 0.5491\\ 0.5491 & 0.3740\end{bmatrix}, U_2=\begin{bmatrix}11.5241 & 4.8428\\ 4.8428 & 6.8908\end{bmatrix}, \Lambda_2=\begin{bmatrix}2.1660 & 1.3128\\ 1.3128 & 0.8953\end{bmatrix},$$

$$E_3=\begin{bmatrix}159.6994 & -14.2791\\ -14.2791 & 149.2812\end{bmatrix}, P_3=\begin{bmatrix}54.4372 & 1.5469\\ 1.5469 & 53.6229\end{bmatrix}, Q_3=\begin{bmatrix}2.2836 & -0.3673\\ -0.3673 & 1.5226\end{bmatrix},$$

$$R_3=\begin{bmatrix}0.2244 & -0.0299\\ -0.0299 & 0.3520\end{bmatrix}, U_3=\begin{bmatrix}5.5253 & -0.1649\\ -0.1649 & 6.7058\end{bmatrix}, \Lambda_3=\begin{bmatrix}0.6446 & -0.01060\\ -0.1060 & 0.9053\end{bmatrix},$$

$$S_1=\begin{bmatrix}-8.4664 & -2.5966\\ -2.5964 & -5.8697\end{bmatrix}, S_2=\begin{bmatrix}-4.4819 & -0.5066\\ -0.5219 & -4.0383\end{bmatrix}, S_3=\begin{bmatrix}-3.3920 & -0.1105\\ -0.0997 & -4.0932\end{bmatrix},$$

The state observer gains are obtained by

$$L_1=\Xi_1^{-1}C_1^{\mathrm{T}}=\begin{bmatrix}0.0197\\ 0.0101\end{bmatrix}, L_2=\Xi_2^{-1}C_2^{\mathrm{T}}=\begin{bmatrix}0.0115\\ 0.0273\end{bmatrix},$$
$$L_3=\Xi_3^{-1}C_3^{\mathrm{T}}=\begin{bmatrix}0.0144\\ 0.0215\end{bmatrix}.$$

Moreover, the error solution $e(t,\varphi)$ of the system satisfies

$$||e(t,\varphi)||\leq 2.3451e^{-0.1t}||\varphi||_h.$$

Figure 1 shows the trajectories of $x_1(t), x_2(t)$, and $x_3(t)$ of the system with the initial conditions $\varphi_1(t)=[1-1]^{\mathrm{T}}$, $\varphi_2(t)=[2-2]^{\mathrm{T}}, \varphi_3(t)=[3-3]^{\mathrm{T}}$. Figure 2 shows the trajectories of the error system $\hat{x}_1(t), \hat{x}_2(t)$, and $\hat{x}_3(t)$.

5. CONCLUSIONS

The problem of full-order observer for nonlinear large-scale systems with interconnected interval time-varying delays has been studied in this article. By introducing a set of augmented Lyapunov–Krasovskii functionals and using a new bounding estimation technique, delay-dependent conditions for designing state observer and exponential stability have been established in terms of LMIs. Furthermore, the LMI-based approach presented in this article provides attractive features in terms of computational efficiency and a straightforward derivation of all the parameters of the observers. A numerical example has been given to illustrate the derived results.

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