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H_∞ control for nonlinear systems with interval non-differentiable time-varying delay

N.T. Thanh [a], V.N. Phat [b,*]

[a] Department of Mathematics, Hanoi University of Mining and Geology, Hanoi, Viet Nam
[b] Institute of Mathematics, VAST, 18 Hoang Quoc Viet Road, Hanoi 10307, Viet Nam

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ABSTRACT

In this paper, we develop a linear matrix inequality approach for studying H_∞ control problem for a class of nonlinear systems with interval time-varying delay. The time delay is assumed to be a continuous function belonging to a given interval, but not necessary to be differentiable. The key features approach includes the construction of new Lyapunov–Krasovskii functionals and the use of a tighter bounding technique. Improved delay-dependent sufficient conditions for the H_∞ control with exponential stability of the system are established in terms of linear matrix inequalities (LMIs). An application to H_∞ control of uncertain linear systems with interval time-varying delay is also given. Numerical examples are given to show the effectiveness of the obtained results.

1. Introduction

Time delays are often encountered in many practical systems because of transmission of the measurement information. The existence of these delays may be the source of instability and serious deterioration in the performance of the closed-loop systems [17,18,15]. In delay-dependent stability analysis of time-delay systems, one concerns to enlarge the feasible region of stability criteria in given time-delay interval, the case of time-varying delay has received less, possibly due to the perceived difficulty of the problem changes in delay make the systems state-space vary with time, which complicates the use of standard analysis tools. In practice, the time-varying delay may vary within an interval where the lower bound is not restricted to being zero. Furthermore, the time derivatives of the time-varying delay can be unknown or undefined. Examples of such systems with an interval time-varying delay are the networked control system, power systems, large-scale systems, economic systems, etc. Since then stability and control of systems containing time-varying delays have been widely studied (see, e.g. [2,14,19,29]).

On the other hand, the H_∞ control of time-delay systems are of practical and theoretical interest since time delay is often encountered in many industrial and engineering processes [3,10,21]. The main objective of the H_∞ control is to obtain a controller that makes the closed-loop system asymptotically stable for a maximum H_∞ performance bound. A significant development in the H_∞ control theory has recently been the introduction of state-space methods [10,26]. This has led to a rather transparent solution to the standard problem of H_∞ control with the objective being to find a feedback controller stabilizing a given system that is subject to some normed suboptimal conditions on perturbations/uncertainties. For the H_∞ control problem, appropriate methods for linear time-delay systems usually make use of the Lyapunov functional approach, whereby the H_∞ conditions are obtained via solving either matrix inequalities or algebraic Riccati-type equations [4,8,22].

More recently, a simple and systematic procedure for constructing time-varying Lyapunov functionals has been studied in [13,12], but for H_∞ filtering. In [16,28,23,25], a modification of the standard LMI-type exponential stability conditions for linear time-delay systems is proposed, allowing to compute two bounds that characterize the exponential nature of the solution in the case of nominal as well as uncertain systems. There the Lyapunov function method was developed to H_∞ control of linear systems with interval time-varying delays, where the assumption on the derivative of the delay function is strictly bounded, but the time-delay function is still assumed to be differentiable. Paper [24] first time studies H_∞ control of linear systems with interval non-differentiable delays, but without the delay in observation and the condition obtained for asymptotic stability. To the best of our knowledge, there has been no investigation on the H_∞ control of delayed systems, where the time delay involved in state and output is interval time-varying and non-differentiable. In fact, this problem is difficult to solve, particularly when the time-varying delays are interval non-differentiable and the output is subjected to such time-varying delay functions. The time

* Corresponding author. Tel.: +84 437563474; fax: +84 437564303.
E-mail address: vnphat@math.ac.vn (V.N. Phat).

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delay is assumed to be any continuous function belonging to a given interval, which means that the lower and upper bounds for the time-varying delay are available, but the delay function is bounded but not necessary to be differentiable. This allows the time-delay to be a fast time-varying function and the lower bound is not restricted to being zero. It is clear that the application of any memoryless feedback controller to such time-delay systems would lead to closed-loop systems with interval time-varying delays. The difficulties then arise when one attempts to derive exponential stabilizability conditions and to extract the controller's parameters for these systems. Indeed, existing Lyapunov–Krasovskii functionals and their associated results in [4,28,25,31,9] cannot be applied to solve the problem posed in this paper as they would either fail to cope with the non-differentiability aspects of the delays, or lead to very complex matrix inequality conditions and any technique such as matrix computation or transformation of variables fails to extract the parameters of the memoryless feedback controllers. This has motivated our research.

In this paper, we develop the results of [28,31,9] for the problem of H_∞ control for nonlinear systems with interval time-varying delay. Compared to the existing results, our result has its own advantages. First, the time delay is assumed to be any continuous function belonging to a given interval, which means that the lower and upper bounds for the time-varying delay are available, but the delay function is bounded but not necessary to be differentiable. Second, both problems of exponential stabilization and H_∞ control will be treated simultaneously. For the former, the controller is required to guarantee the global exponential stability for the closed-loop system. As to the latter, a prescribed performance in an H_∞ sense is also required to be achieved for all admissible uncertainties. By constructing a set of improved Lyapunov Krasovskii functionals and using new bounding estimation technique, a new delay-dependent condition for the robust H_∞ control is established in terms of LMIs, that can be solved numerically in an efficient manner by using standard computational algorithms [5]. The approach allows us to apply to H_∞ control of uncertain linear systems with interval non-differentiable time-varying delay.

The paper is organized as follows. Section 2 presents definitions and some well-known technical propositions needed for the proof of the main results. H_∞ controller design for exponential stability and an application to uncertain linear systems with non-differentiable time-varying delay are presented in Section 3. Numerical examples showing the effectiveness of the obtained results are given in Section 4. The paper ends with conclusions.

2. Preliminaries

The following notations will be used throughout this paper, R^+ denotes the set of all real-negative numbers; R^n denotes the n-dimensional space with the scalar product $(.,.)$ and the vector norm $\|\cdot\|$; $R^{n\times r}$ denotes the space of all matrices of $(n\times r)$- dimension. A^T denotes the transpose of A; a matrix A is symmetric if $A=A^T$; I denotes the identity matrix; $\lambda(A)$ denotes all the eigenvalues of A; $\lambda_{max}(A)=\max\{\text{Re}\,\lambda:\ \lambda\in\lambda(A)\}$; $\lambda_{min}(A)=\min\{\text{Re}\,\lambda:\ \lambda\in\lambda(A)\}$; $C^1([-\tau,0], R^n)$ denotes the set of all R^n-valued differentiable functions on $[-\tau,0]$; $L_2([0,\infty],R^r)$ stands for the set of all square-integrable R^r-valued functions on $[0,\infty]$. The symmetric terms in a matrix are denoted by $*$. Matrix A is semi-positive definite $(A\geq 0)$ if $(Ax,x)\geq 0$, for all $x\in R^n$; A is positive definite $(A>0)$ if $(Ax,x)>0$ for all $x\neq 0$; $A\geq B$ means $A-B\geq 0$. The following norms will be used: $\|\cdot\|$ refers to the Euclidean vector norm; $\|\varphi\|_C=\sup_{t\in[-\tau,0]}\|\varphi(t)\|$ stands for the norm of a function $\varphi(\cdot)\in C([-\tau,0],R^n)$. The segment of the trajectory $x(t)$ is denoted by $x_t=\{x(t+s):\ s\in[-\tau,0]\}$ with its norm $\|x_t\|=\sup_{s\in[-\tau,0]}\|x(t+s)\|$.

Consider a system with time-varying delay of the form

$$\begin{cases}\dot{x}(t)=Ax(t)+Dx(t-h(t))+Bu(t)+C\omega(t)+f(t,x(t),x(t-h(t)),u(t),\omega(t)),\\ z(t)=Ex(t)+Gx(t-h(t))+Fu(t)+g(t,x(t),x(t-h(t)),u(t)),\end{cases} \tag{2.1}$$

with the initial conditions

$$x(t_0+\theta)=\varphi(\theta),\quad \forall\theta\in\mathcal{I}_{t_0,h},$$
$$(t_0,\varphi)\in R^+\times C([-\tau,0],R^n),$$

where $\varphi:\mathcal{I}_{t_0,h}\to R^n$ is a continuous norm-bounded initial condition (see also [18,19]) and

$$\mathcal{I}_{t_0,h}=\{t\in R:\ t=\eta-h(\eta)\leq t_0,\ \eta\geq t_0\};\quad \tau=\sup_{t_0\in R^+,\ t\in\mathcal{I}_{t_0,h}}(t_0-t);$$

$h(t):R^+\to R^+$ is a continuous function satisfying $0\leq h_1\leq h(t)\leq h_2$, $\forall t\geq 0$. We see in this case that $h_2=\tau$.

$x(\cdot)\in R^n$ is the state vector; $u(\cdot)\in L_2([0,s],R^m)$, $s\geq 0$, is the control vector; $\omega(\cdot)\in L_2([0,\infty],R^r)$ is the uncertainty input; $z(t)\in R^s$ is the observation output.

Let $x^h:=x(t-h(t))$, the nonlinear functions $f(t,x,x^h,u,\omega):R^+\times R^n\times R^n\times R^m\times R^r\to R^n$, $g(t,x,x^h,u):\ R^+\times R^n\times R^n\times R^m\to R^s$ satisfy the following growth conditions:

$$\exists a,b,c,d>0:\ \|f(t,x,x^h,u,\omega)\|\leq a\|x\|+b\|x^h\|+c\|u\|+d\|\omega\|,\forall(x,x^h,u,\omega)$$

$$\exists a_1,b_1,c_1>0:\ \|g(t,x,x^h,u)\|^2\leq a_1\|x\|^2+b_1\|x^h\|^2+c_1\|u\|^2,\forall(x,x^h,u). \tag{2.2}$$

We assume $\varphi(\cdot)\in C^1([-\tau,0],R^n)$ and $\|\varphi\|_{C_1}=\sup_{t\in[-\tau,0]}\|\varphi(t)\|+\sup_{t\in[-\tau,0]}\|\dot{\varphi}(t)\|$ stands for the norm of a function $\varphi(\cdot)\in C^1([-\tau,0],R^n)$. Once the above assumption on $\varphi(\cdot),f(\cdot)$ are given, the solution of system (2.1) is well defined (see, e.g. [18,15]).

Definition 1. Given $\beta>0$. The zero solution of system (2.1), where $u(t)=0$, $\omega(t)=0$, is β–stable if there is a positive number $N>0$ such that every solution of the system satisfies

$$\|x(t,\varphi)\|\leq N\|\varphi\|_{C_1}e^{-\beta t},\quad \forall t\geq 0.$$

Definition 2. Given $\beta>0$, $\gamma>0$. The H_∞ control problem for system (2.1) has a solution if there exists a memoryless state feedback controller $u(t)=Yx(t)$ satisfying the following two requirements:

(i) The zero solution of the closed-loop system, where $\omega(t)=0$,

$$\dot{x}(t)=[A+BY]x(t)+Dx(t-h(t))+f(t,x,x^h,u,0), \tag{2.3}$$

is β–stable.

(ii) There is a number $c_0>0$ such that

$$\sup\frac{\int_0^\infty\|z(t)\|^2\,dt}{c_0\|\varphi\|^2_{C_1}+\int_0^\infty\|\omega(t)\|^2dt}\leq\gamma,$$

where the supremum is taken over all $\varphi\in C^1([-\tau,0],R^n)$ and the non-zero uncertainty $\omega(t)\in L_2([0,\infty],R^r)$. In this case we say that the feedback control $u(t)=Yx(t)$ exponentially stabilizes the system.

Proposition 1. *Let P,Q be matrices of appropriate dimensions and Q is symmetric positive definite. Then,*

$$(2Px,y)-(Qy,y)\leq(PQ^{-1}P^Tx,x).$$

The proof of the above proposition is easily derived from completing the square

$$0\leq(Q(y-Q^{-1}P^Tx),y-Q^{-1}P^Tx).$$

Proposition 2 (*Schur complement lemma, Boyd et al. [1]*). *Given constant matrices X,Y,Z, where $Y=Y^T>0$. Then $X+Z^TY^{-1}Z<0$ if and only if*

$$\begin{bmatrix} X & Z^T \\ Z & -Y \end{bmatrix} < 0.$$

Proposition 3 (*Sun et al. [27]*). *For any constant matrix $Z=Z^T>0$ and positive numbers $h,\bar{h}$ such that the following integrals are well defined, then*

(i)

$$-\int_{t-h}^{t} x(s)^T Z x(s)\, ds \le -\frac{1}{h}\left(\int_{t-h}^{t} x(s)\, ds\right)^T Z\left(\int_{t-h}^{t} x(s)\, ds\right).$$

(ii)

$$-\int_{-\bar{h}}^{-h}\int_{t+s}^{t} x(\tau)^T Z x(\tau)\, d\tau\, ds$$

$$\le -\frac{2}{\bar{h}^2-h^2}\left(\int_{-\bar{h}}^{-h}\int_{t+s}^{t} x(\tau)\, d\tau\, ds\right)^T Z\left(\int_{-\bar{h}}^{-h}\int_{t+s}^{t} x(\tau)\, d\tau\, ds\right).$$

Proposition 4 (*Lower bounds lemma, Park et al. [20]*). *Let $f_1,f_2,\ldots,f_N:\ R^m \to R$ have positive values in an open subset D of R^m. Then, the reciprocally convex combination of f_i over D satisfies*

$$\min_{\{r_i | r_i>0, \sum_i r_i=1\}} \sum_i \frac{1}{r_i} f_i(t) = \sum_i f_i(t) + \max_{g_{i,j}(t)} \sum_{i\ne j} g_{i,j}(t)$$

subject to

$$\left\{ g_{i,j}:\ R^m \to R, g_{j,i}(t)=g_{i,j}(t), \begin{bmatrix} f_i(t) & g_{i,j}(t) \\ g_{i,j}(t) & f_j(t) \end{bmatrix} \ge 0 \right\}.$$

3. Main result

In this section, we first design H_∞ controller for system (2.1) with interval time-varying delays. By constructing a new set of Lyapunov–Krasovskii functionals, a new delay-dependent exponential stability criterion for the closed-loop system with a delay varying in an interval is derived in terms of LMIs. Then we give an application to H_∞ control of uncertain linear systems with interval time-varying delay. Before introducing the main result, the following notations of several matrix variables are defined for simplicity:

$$\tau = h_2, \quad \varepsilon = a+b+c+\frac{4d^2}{\gamma},$$

$$P_1 = P^{-1}, \quad Q_1 = P^{-1}QP^{-1}, \quad R_1 = P^{-1}RP^{-1},$$

$$U_1 = P^{-1}UP^{-1}, \quad \Lambda_1 = P^{-1}\Lambda^{-1}P^{-1}, \quad S_1 = P^{-1}SP^{-1}$$

$$\Omega_{11} = \begin{bmatrix} T_{11} & T_{12} & T_{13} & T_{14} & T_{15} & T_{16} & 0 & 0 \\ * & T_{22} & T_{23} & T_{24} & T_{25} & T_{26} & 0 & 0 \\ * & * & T_{33} & T_{34} & T_{35} & T_{36} & 0 & 0 \\ * & * & * & T_{44} & T_{45} & T_{46} & 0 & 0 \\ * & * & * & * & T_{55} & T_{56} & 0 & 0 \\ * & * & * & * & * & T_{66} & 0 & 0 \\ * & * & * & * & * & * & -U & -S \\ * & * & * & * & * & * & * & -U \end{bmatrix}$$

$$\Omega_{22} = \begin{bmatrix} -\frac{I}{3} & 0 & 0 & 0 & 0 \\ * & -\frac{I}{3} & 0 & 0 & 0 \\ * & * & -\frac{I}{4a_1+2a} & 0 & 0 \\ * & * & * & -\frac{I}{4b_1+2b} & 0 \\ * & * & * & * & -\frac{I}{2+4c_1+2c} \end{bmatrix},$$

$$\Omega_{12} = \begin{bmatrix} PE^T & 0 & P & 0 & Y^T \\ 0 & PG^T & 0 & P & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$T_{11} = AP + PA^T + 2\beta P - R(e^{-2\beta h_1} + e^{-2\beta h_2}) + \frac{4}{\gamma}CC^T + \varepsilon I - \frac{2e^{-4\beta h_2}(h_2-h_1)}{h_2+h_1}\Lambda + (BY + Y^TB^T) + 2Q,$$

$$T_{12} = DP, \quad T_{13} = e^{-2\beta h_1}R, \quad T_{14} = e^{-2\beta h_2}R$$

$$T_{15} = \frac{2e^{-4\beta h_2}(h_2-h_1)}{h_2+h_1}\Lambda, \quad T_{16} = PA^T + BY,$$

$$T_{22} = -2e^{-2\beta h_2}U + e^{-2\beta h_2}(S+S^T),$$

$$T_{23} = e^{-2\beta h_2}U - e^{-2\beta h_2}S, \quad T_{24} = e^{-2\beta h_2}U - e^{-2\beta h_2}S^T, \quad T_{25} = 0,$$

$$T_{26} = PD^T$$

$$T_{33} = -e^{-2\beta h_1}Q - e^{-2\beta h_1}R - e^{-2\beta h_2}U,$$

$$T_{34} = e^{-2\beta h_2}S^T, \quad T_{35} = 0, \quad T_{36} = 0,$$

$$T_{44} = -e^{-2\beta h_2}Q - e^{-2\beta h_2}R - e^{-2\beta h_2}U, \quad T_{45} = 0, \quad T_{46} = 0,$$

$$T_{55} = -\frac{2e^{-4\beta h_2}}{h_2^2-h_1^2}\Lambda, \quad T_{56} = 0,$$

$$T_{66} = R(h_1^2 + h_2^2) + (h_2-h_1)^2U + h_2(h_2-h_1)\Lambda - 2P + \frac{4}{\gamma}CC^T + \varepsilon I.$$

$$\alpha_1 = \lambda_{min}(P_1),$$

$$\alpha_2 = \lambda_{max}(P_1) + \beta^{-1}\lambda_{max}(Q_1) + h_1^3\lambda_{max}(R_1) + h_2^3\lambda_{max}(R_1) + (h_2-h_1)^3\lambda_{max}(U_1) + (h_2-h_1)h_2^2\lambda_{max}(\Lambda_1).$$

For simplicity of expression as in [21,26], we assume that

$$F^T[E,G] = 0, \quad F^TF = I. \tag{3.1}$$

The following is the main result of the paper, which gives sufficient conditions for the H_∞ control of system (2.1). Essentially, the proof is based on the construction of improved Lyapunov–Krasovskii functionals satisfying Lyapunov stability theorem for the time-delay system [6].

Theorem 1. *The H_∞ control of system (2.1) has a solution if there exist symmetric positive definite matrices P,Q,R,U,Λ and matrices Y,S such that the following LMI holds:*

$$\begin{bmatrix} \Omega_{11} & \Omega_{12} \\ * & \Omega_{22} \end{bmatrix} < 0. \tag{3.2}$$

Moreover, stabilizing feedback control is given by

$$u(t) = YP^{-1}x(t), \quad t \ge 0,$$

and the solution of the system satisfies

$$\|x(t,\varphi)\| \le \sqrt{\frac{\alpha_2}{\alpha_1}}\|\varphi\|_{C_1} e^{-\beta t}, \quad t \ge 0.$$

Proof. Consider the following Lyapunov–Krasovskii functional for the closed-loop system:

$$V(t,x_t) = \sum_{i=1}^{7} V_i(t,x_t),$$

where

$$V_1(t,x_t)=x(t)^TP_1x(t),\quad V_2(t,x_t)=\int_{t-h_1}^{t}e^{2\beta(s-t)}x(s)^TQ_1x(s)\,ds,$$
$$V_3(t,x_t)=\int_{t-h_2}^{t}e^{2\beta(s-t)}x(s)^TQ_1x(s)\,ds,$$
$$V_4(t,x_t)=h_1\int_{-h_1}^{0}\int_{t+s}^{t}e^{2\beta(\tau-t)}\dot{x}(\tau)^TR_1\dot{x}(\tau)\,d\tau\,ds,$$
$$V_5(t,x_t)=h_2\int_{-h_2}^{0}\int_{t+s}^{t}e^{2\beta(\tau-t)}\dot{x}(\tau)^TR_1\dot{x}(\tau)\,d\tau\,ds,$$
$$V_6(t,x_t)=(h_2-h_1)\int_{-h_2}^{-h_1}\int_{t+s}^{t}e^{2\beta(\tau-t)}\dot{x}(\tau)^TU_1\dot{x}(\tau)\,d\tau\,ds,$$
$$V_7(t,x_t)=\int_{-h_2}^{-h_1}\int_{\theta}^{0}\int_{t+s}^{t}e^{2\beta(\tau+s-t)}\dot{x}(\tau)^T\Lambda_1\dot{x}(\tau)\,d\tau\,ds\,d\theta.$$

It is easy to verify that

$$\alpha_1\|x(t)\|^2\le V(t,x_t),\ t\in R^+\quad\text{and}\quad V(0,x_0)\le\alpha_2\|\varphi\|^2_{C_1}. \tag{3.3}$$

Taking the derivative of $V_i(\cdot)$ in t along the solution of the system, we obtain

$$\begin{aligned}\dot{V}_1(t,x_t)&=2x(t)^TP_1\dot{x}(t)\\&=2x(t)^TP_1[Ax(t)+Dx(t-h(t))+Bu(t)+C\omega(t)+f(\cdot)]\\&=2x(t)^TP_1Ax(t)+2x(t)^TP_1Dx(t-h(t))+2x(t)^TP_1Bu(t)\\&\quad+2x(t)^TP_1C\omega(t)+2x(t)^TP_1f(\cdot)+2\beta x(t)^TP_1x(t)-2\beta V_1(t,x_t)\end{aligned}$$

$$\dot{V}_2(t,x_t)=x(t)^TQ_1x(t)-e^{-2\beta h_1}x(t-h_1)^TQ_1x(t-h_1)-2\beta V_2(t,x_t),$$
$$\dot{V}_3(t,x_t)=x(t)^TQ_1x(t)-e^{-2\beta h_2}x(t-h_2)^TQ_1x(t-h_2)-2\beta V_3(t,x_t),$$
$$\dot{V}_4(t,x_t)\le h_1^2\dot{x}(t)^TR_1\dot{x}(t)-h_1e^{-2\beta h_1}\int_{t-h_1}^{t}\dot{x}(s)^TR_1\dot{x}(s)\,ds-2\beta V_4(t,x_t)$$
$$\dot{V}_5(t,x_t)\le h_2^2\dot{x}(t)^TR_1\dot{x}(t)-h_2e^{-2\beta h_2}\int_{t-h_2}^{t}\dot{x}(s)^TR_1\dot{x}(s)\,ds-2\beta V_5(t,x_t)$$
$$\dot{V}_6(t,x_t)\le(h_2-h_1)^2\dot{x}(t)^TU_1\dot{x}(t)-(h_2-h_1)e^{-2\beta h_2}\int_{t-h_2}^{t-h_1}\dot{x}(s)^TU_1\dot{x}(s)\,ds-2\beta V_6(t,x_t),$$
$$\begin{aligned}\dot{V}_7(t,x_t)&\le(h_2-h_1)h_2\dot{x}(t)^T\Lambda_1\dot{x}(t)\\&\quad-e^{-4\beta h_2}\int_{-h_2}^{-h_1}\int_{t+\theta}^{t}\dot{x}(s)^T\Lambda_1\dot{x}(s)\,ds\,d\theta-2\beta V_7(t,x_t)\end{aligned}$$

Applying Proposition 3 and the Newton–Leibniz formula

$$\int_{t-h_i}^{t}\dot{x}(s)\,ds=x(t)-x(t-h_i),\quad i=1,2,$$

we have

$$\begin{aligned}-h_i\int_{t-h_i}^{t}\dot{x}^T(s)R_1\dot{x}(s)\,ds&\le-\left[\int_{t-h_i}^{t}\dot{x}(s)\,ds\right]^TR_1\left[\int_{t-h_i}^{t}\dot{x}(s)\,ds\right]\\&\le-[x(t)-x(t-h_i)]^TR_1[x(t)-x(t-h_i)]\end{aligned}$$

Note that

$$\int_{t-h_2}^{t-h_1}\dot{x}^T(s)U_1\dot{x}(s)\,ds=\int_{t-h_2}^{t-h(t)}\dot{x}^T(s)U_1\dot{x}(s)\,ds+\int_{t-h(t)}^{t-h_1}\dot{x}^T(s)U_1\dot{x}(s)\,ds.$$

Using Proposition 3 gives

$$\begin{aligned}[h_2-h(t)]\int_{t-h_2}^{t-h(t)}\dot{x}^T(s)U_1\dot{x}(s)\,ds&\ge\left[\int_{t-h_2}^{t-h(t)}\dot{x}(s)\,ds\right]^TU_1\left[\int_{t-h_2}^{t-h(t)}\dot{x}(s)\,ds\right]\\&\ge[x(t-h(t))-x(t-h_2)]^TU_1[x(t-h(t))-x(t-h_2)]\end{aligned}$$

Since $h_2-h(t)\le h_2-h_1$, we have

$$\begin{aligned}(h_2-h_1)\int_{t-h_2}^{t-h(t)}\dot{x}^T(s)U_1\dot{x}(s)\,ds&\ge\frac{h_2-h_1}{h_2-h(t)}[x(t-h(t))\\&\quad-x(t-h_2)]^TU_1[x(t-h(t))-x(t-h_2)],\end{aligned}$$

then

$$\begin{aligned}-(h_2-h_1)\int_{t-h_2}^{t-h(t)}\dot{x}^T(s)U_1\dot{x}(s)\,ds&\le-\frac{h_2-h_1}{h_2-h(t)}[x(t-h(t))-x(t-h_2)]^T\\&\quad U_1[x(t-h(t))-x(t-h_2)].\end{aligned}$$

Similarly, we have

$$\begin{aligned}-(h_2-h_1)\int_{t-h(t)}^{t-h_1}\dot{x}^T(s)U_1\dot{x}(s)\,ds&\le-\frac{h_2-h_1}{h(t)-h_1}[x(t-h_1)-x(t-h(t))]^T\\&\quad U_1[x(t-h_1)-x(t-h(t))].\end{aligned}$$

On the other hand, the condition (3.2) figures out $\begin{bmatrix}U & S\\ S^T & U\end{bmatrix}\ge0$, so using Proposition 4 with $r_1=(h_2-h(t))/(h_2-h_1)$; $r_2=(h(t)-h_1)/(h_2-h_1)$ gives the following inequalities:

$$\begin{bmatrix}\sqrt{\frac{r_2}{r_1}}[x(t-h(t))-x(t-h_2))]\\-\sqrt{\frac{r_1}{r_2}}[x(t-h_1)-x(t-h(t))]\end{bmatrix}^T\begin{bmatrix}U_1 & S_1\\ S_1^T & U_1\end{bmatrix}\begin{bmatrix}\sqrt{\frac{r_2}{r_1}}[x(t-h(t))-x(t-h_2))]\\-\sqrt{\frac{r_1}{r_2}}[x(t-h_1)-x(t-h(t))]\end{bmatrix}$$
$$=\begin{bmatrix}\sqrt{\frac{r_2}{r_1}}[x(t-h(t))-x(t-h_2))]\\-\sqrt{\frac{r_1}{r_2}}[x(t-h_1)-x(t-h(t))]\end{bmatrix}^T\begin{bmatrix}P_1 & 0\\ 0 & P_1\end{bmatrix}\begin{bmatrix}U & S\\ S^T & U\end{bmatrix}$$
$$\begin{bmatrix}P_1 & 0\\ 0 & P_1\end{bmatrix}\begin{bmatrix}\sqrt{\frac{r_2}{r_1}}[x(t-h(t))-x(t-h_2))]\\-\sqrt{\frac{r_1}{r_2}}[x(t-h_1)-x(t-h(t))]\end{bmatrix}\ge0,$$

equivalently,

$$\begin{aligned}&-\frac{r_2}{r_1}[x(t-h(t))-x(t-h_2)]^TU_1[x(t-h(t))-x(t-h_2)]\\&\quad-\frac{r_1}{r_2}[x(t-h_1)-x(t-h(t))]^TU_1[x(t-h_1)-x(t-h(t))]\\&\quad\le-[x(t-h(t))-x(t-h_2)]^TS_1[x(t-h_1)-x(t-h(t))]\\&\quad-[x(t-h_1)-x(t-h(t))]^TS_1^T[x(t-h(t))-x(t-h_2)],\end{aligned}$$

and

$$\begin{aligned}&-(h_2-h_1)\int_{t-h_2}^{t-h_1}\dot{x}^T(s)U_1\dot{x}(s)\,ds\\&\quad\le-\frac{h_2-h_1}{h_2-h(t)}[x(t-h(t))-x(t-h_2)]^TU_1[x(t-h(t))-x(t-h_2)]\\&\qquad-\frac{h_2-h_1}{h(t)-h_1}[x(t-h_1)-x(t-h(t))]^TU_1[x(t-h_1)-x(t-h(t))]\\&\quad=-\frac{1}{r_1}[x(t-h(t))-x(t-h_2)]^TU_1[x(t-h(t))-x(t-h_2)]\\&\qquad-\frac{1}{r_2}[x(t-h_1)-x(t-h(t))]^TU_1[x(t-h_1)-x(t-h(t))]\\&\quad\le-[x(t-h(t))-x(t-h_2)]^TU_1[x(t-h(t))-x(t-h_2)]\\&\qquad-[x(t-h_1)-x(t-h(t))]^TU_1[x(t-h_1)-x(t-h(t))]\\&\qquad-[x(t-h(t))-x(t-h_2)]^TS_1[x(t-h_1)-x(t-h(t))]\\&\qquad-[x(t-h_1)-x(t-h(t))]^TS_1^T[x(t-h(t))-x(t-h_2)].\end{aligned}$$

Therefore, we have

$$\dot{V}_4(t,x_t)\le h_1^2\dot{x}(t)^TR_1\dot{x}(t)-e^{-2\beta h_1}[x(t)-x(t-h_1)]^TR_1[x(t)-x(t-h_1)]-2\beta V_4(t,x_t),$$
$$\begin{aligned}\dot{V}_5(t,x_t)&\le h_2^2\dot{x}(t)^TR_1\dot{x}(t)-e^{-2\beta h_2}[x(t)-x(t-h_2)]^TR_1[x(t)-x(t-h_2)]\\&\quad-2\beta V_5(t,x_t),\end{aligned}$$

$$\dot V_6(t,x_t) \le (h_2-h_1)^2\dot x(t)^T U_1\dot x(t) - 2\beta V_6(t,x_t)$$
$$-e^{-2\beta h_2}[x(t-h(t))-x(t-h_2)]^T U_1[x(t-h(t))-x(t-h_2)]$$
$$-e^{-2\beta h_2}[x(t-h_1)-x(t-h(t))]^T U_1[x(t-h_1)-x(t-h(t))]$$
$$-e^{-2\beta h_2}[x(t-h(t))-x(t-h_2)]^T S_1[x(t-h_1)-x(t-h(t))]$$
$$-e^{-2\beta h_2}[x(t-h_1)-x(t-h(t))]^T S_1^T[x(t-h(t))-x(t-h_2)] \quad (3.4)$$

Note that when $h(t)=h_1$ or $h(t)=h_2$, the relation (3.4) still holds. Besides, using the Proposition 3 again, we have

$$-e^{-4\beta h_2}\int_{-h_2}^{-h_1}\int_{t+\theta}^{t}\dot x(s)^T\Lambda_1\dot x(s)\,ds\,d\theta$$
$$\le -e^{-4\beta h_2}\frac{2}{h_2^2-h_1^2}\left(\int_{-h_2}^{-h_1}\int_{t+\theta}^{t}\dot x(s)\,dsd\theta\right)^T\Lambda_1\left(\int_{-h_2}^{-h_1}\int_{t+\theta}^{t}\dot x(s)\,dsd\theta\right)$$
$$\le -\frac{2e^{-4\beta h_2}}{h_2^2-h_1^2}\left((h_2-h_1)x(t)-\int_{t-h_2}^{t-h_1}x(\theta)\,d\theta\right)^T\Lambda_1\left((h_2-h_1)x(t)-\int_{t-h_2}^{t-h_1}x(\theta)\,d\theta\right).$$

Hence,

$$\dot V_7(\cdot) \le (h_2-h_1)h_2\dot x(t)^T\Lambda_1\dot x(t) - 2\beta V_7(t,x_t)$$
$$-\frac{2e^{-4\beta h_2}}{h_2^2-h_1^2}\left((h_2-h_1)x(t)-\int_{t-h_2}^{t-h_1}x(\theta)\,d\theta\right)^T\Lambda_1\left((h_2-h_1)x(t)-\int_{t-h_2}^{t-h_1}x(\theta)\,d\theta\right) \quad (3.5)$$

From the following identity relation:

$$-2\dot x(t)^T P_1[\dot x(t)-Ax(t)-Dx(t-h(t))-Bu(t)-C\omega(t)-f(\cdot)]=0,$$

combining the condition (2.2) and applying Proposition 1 for the following inequalities:

$$2x^T P_1 f(t,x,x^h,u,\omega) \le 2\|P_1x\|\|f(t,x,x^h,u,\omega)\|$$
$$\le 2\|P_1x\|[a\|x\|+b\|x^h\|+c\|u\|+d\|\omega\|]$$
$$\le a\|P_1x\|^2+a\|x\|^2+b\|P_1x\|^2+b\|x^h\|^2$$
$$+c\|P_1x\|^2+c\|u\|^2+\frac{4d^2}{\gamma}\|P_1x\|^2+\frac{\gamma}{4}\|\omega\|^2$$
$$=a\|x\|^2+b\|x^h\|^2+c\|u\|^2+\frac{\gamma}{4}\|\omega\|^2+\varepsilon\|P_1x\|^2$$

$$2\dot x(t)^T P_1 f(t,x,x^h,u,\omega) \le a\|x\|^2+b\|x^h\|^2+c\|u\|^2+\frac{\gamma}{4}\|\omega\|^2+\varepsilon\|P_1\dot x(t)\|^2,$$
$$2x(t)^T P_1C\omega \le \frac{\gamma}{4}\|\omega\|^2+\frac{4}{\gamma}x(t)^T P_1CC^T P_1x(t)$$
$$2\dot x(t)^T P_1C\omega \le \frac{\gamma}{4}\|\omega\|^2+\frac{4}{\gamma}\dot x(t)^T P_1CC^T P_1\dot x(t),$$

we obtain

$$0=-2\dot x(t)^T P_1[\dot x(t)-Ax(t)-Dx(t-h(t))-Bu(t)-C\omega(t)-f(\cdot)]$$
$$\le -2\dot x(t)^T P_1[\dot x(t)-Ax(t)-Dx(t-h(t))-BYP_1x(t)]$$
$$+2\dot x(t)^T P_1C\omega(t)+2\dot x(t)^T P_1 f(\cdot)$$
$$\le -2\dot x(t)^T P_1[\dot x(t)-Ax(t)-Dx(t-h(t))-BYP_1x(t)]$$
$$+\frac{\gamma}{4}\|\omega\|^2+\frac{4}{\gamma}\dot x(t)^T P_1CC^T P_1\dot x(t)$$
$$+a\|x\|^2+b\|x^h\|^2+c\|u\|^2+\frac{\gamma}{4}\|\omega\|^2+\varepsilon\|P_1\dot x(t)\|^2, \quad (3.6)$$

and

$$\dot V_1(\cdot)+2\beta V_1(\cdot)=2\beta x(t)^T P_1x(t)+2x(t)^T P_1Ax(t)+2x(t)^T P_1Dx(t-h(t))$$
$$+2x(t)^T P_1BYP_1x(t)+2x(t)^T P_1C\omega(t)+2x(t)^T P_1 f(\cdot)$$
$$\le 2\beta x(t)^T P_1x(t)+2x(t)^T P_1Ax(t)+2x(t)^T P_1Dx(t-h(t))+2x(t)^T P_1BYP_1x(t)$$
$$+\frac{\gamma}{4}\|\omega\|^2+\frac{4}{\gamma}x(t)^T P_1CC^T P_1x(t)$$
$$+a\|x\|^2+b\|x^h\|^2+c\|u\|^2+\frac{\gamma}{4}\|\omega\|^2+\varepsilon\|P_1x\|^2. \quad (3.7)$$

Moreover, applying the inequalities (3.4)–(3.7) leads to

$$\dot V(\cdot)+2\beta V(\cdot) \le x(t)^T\Big[P_1A+A^TP_1+2\beta P_1+P_1(BY+Y^TB^T)P_1+2Q_1$$
$$-R_1(e^{-2\beta h_1}+e^{-2\beta h_2})-\frac{2e^{-4\beta h_2}(h_2-h_1)}{h_2+h_1}\Lambda_1+2aI+2cP_1Y^TYP_1$$
$$+\frac{4}{\gamma}P_1CC^TP_1+\varepsilon P_1^2\Big]x(t)$$
$$+2x(t)^T[P_1D]x(t-h(t))+2x(t)^T[e^{-2\beta h_1}R_1]x(t-h_1)$$
$$+2x(t)^T[e^{-2\beta h_2}R_1]x(t-h_2)+2x(t)^T\left[\frac{2e^{-4\beta h_2}(h_2-h_1)}{h_2+h_1}\Lambda_1\right]\int_{t-h_2}^{t-h_1}x(\theta)\,d\theta$$
$$+2x(t)^T[A^TP_1+P_1BYP_1]\dot x(t)$$
$$+x(t-h(t))^T[-2e^{-2\beta h_2}U_1+2bI+e^{-2\beta h_2}(S_1+S_1^T)]x(t-h(t))$$
$$+2x(t-h(t))^T[e^{-2\beta h_2}U_1-e^{-2\beta h_2}S_1]x(t-h_1)$$
$$+2x(t-h(t))^T[e^{-2\beta h_2}U_1-e^{-2\beta h_2}S_1^T]x(t-h_2)$$
$$+2x(t-h(t))^T[0]\int_{t-h_2}^{t-h_1}x(\theta)\,d\theta+2x(t-h(t))^T[D^TP_1]\dot x(t)$$
$$+x(t-h_1)^T[-e^{-2\beta h_1}Q_1-e^{-2\beta h_1}R_1-e^{-2\beta h_2}U_1]x(t-h_1)$$
$$+2x(t-h_1)^Te^{-2\beta h_2}S_1^Tx(t-h_2)+2x(t-h_1)^T[0]\int_{t-h_2}^{t-h_1}x(\theta)\,d\theta$$
$$+2x(t-h_1)^T[0]\dot x(t)$$
$$+2x(t-h_2)^T[0]\int_{t-h_2}^{t-h_1}x(\theta)\,d\theta+x(t-h_2)^T[-e^{-2\beta h_2}Q_1-e^{-2\beta h_2}R_1$$
$$-e^{-2\beta h_2}U_1]x(t-h_2)$$
$$+2x(t-h_2)^T[0]\dot x(t)+\gamma\|\omega(t)\|^2$$
$$+\left(\int_{t-h_2}^{t-h_1}x(\theta)\,d\theta\right)^T\left[-\frac{2e^{-4\beta h_2}}{h_2^2-h_1^2}\Lambda_1\right]\left(\int_{t-h_2}^{t-h_1}x(\theta)\,d\theta\right)$$
$$+2\left(\int_{t-h_2}^{t-h_1}x(\theta)\,d\theta\right)^T[0]\dot x(t)+\dot x(t)^T[R_1(h_1^2+h_2^2)+(h_2-h_1)^2U_1$$
$$+h_2(h_2-h_1)\Lambda_1-2P_1+\frac{4}{\gamma}P_1CC^TP_1+\varepsilon P_1^2]\dot x(t)$$

Setting $y(t)=P^{-1}x(t)$ or $x(t)=Py(t)$, we obtain

$$\dot V(\cdot)+2\beta V(\cdot) \le y(t)^T\Big[AP+PA^T+2\beta P+(BY+Y^TB^T)+2Q-R(e^{-2\beta h_1}+e^{-2\beta h_2})$$
$$-\frac{2e^{-4\beta h_2}(h_2-h_1)}{h_2+h_1}\Lambda+2aP^2+2cY^TY+\frac{4}{\gamma}CC^T+\varepsilon I\Big]y(t)$$
$$+2y(t)^T[DP]y(t-h(t))+2y(t)^T[e^{-2\beta h_1}R]y(t-h_1)$$
$$+2y(t)^T[e^{-2\beta h_2}R]y(t-h_2)+2y(t)^T\left[\frac{2e^{-4\beta h_2}(h_2-h_1)}{h_2+h_1}\Lambda\right]\int_{t-h_2}^{t-h_1}y(\theta)\,d\theta$$
$$+2y(t)^T[PA^T+BY]\dot y(t)$$
$$+y(t-h(t))^T[-2e^{-2\beta h_2}U+2bP^2+e^{-2\beta h_2}(S+S^T)]y(t-h(t))$$
$$+2y(t-h(t))^T[e^{-2\beta h_2}U-e^{-2\beta h_2}S]y(t-h_1)$$
$$+2y(t-h(t))^T[e^{-2\beta h_2}U-e^{-2\beta h_2}S^T]y(t-h_2)$$
$$+2y(t-h(t))^T[0]\int_{t-h_2}^{t-h_1}y(\theta)\,d\theta+2y(t-h(t))^T[PD^T]\dot y(t)$$
$$+y(t-h_1)^T[-e^{-2\beta h_1}Q-e^{-2\beta h_1}R-e^{-2\beta h_2}U]y(t-h_1)$$
$$+2y(t-h_1)^T[0]\int_{t-h_2}^{t-h_1}y(\theta)\,d\theta+2y(t-h_1)^T[0]\dot y(t)$$
$$+2y(t-h_1)^Te^{-2\beta h_2}S^Ty(t-h_2)$$
$$+y(t-h_2)^T[-e^{-2\beta h_2}Q-e^{-2\beta h_2}R-e^{-2\beta h_2}U]y(t-h_2)$$
$$+2y(t-h_2)^T[0]\int_{t-h_2}^{t-h_1}y(\theta)\,d\theta+2y(t-h_2)^T[0]\dot y(t)+\gamma\|\omega(t)\|^2.$$
$$+\left(\int_{t-h_2}^{t-h_1}y(\theta)\,d\theta\right)^T\left[-\frac{2e^{-4\beta h_2}}{h_2^2-h_1^2}\Lambda\right]\left(\int_{t-h_2}^{t-h_1}y(\theta)\,d\theta\right)$$

$$+2\left(\int_{t-h_2}^{t-h_1} y(\theta)\,d\theta\right)^T[0]\dot{y}(t)$$
$$+\dot{y}(t)^T\left[R(h_1^2+h_2^2)+(h_2-h_1)^2U+h_2(h_2-h_1)\Lambda-2P+\frac{4}{\gamma}CC^T+\varepsilon I\right]\dot{y}(t)$$

Hence, we have

$$\begin{aligned}\dot{V}(\cdot)+2\beta V(\cdot)\le&\gamma\|\omega(t)\|^2+\xi(t)^TM\xi(t)\\&-y(t)^T[3PE^TEP+4a_1P^2+(2+4c_1)Y^TY]y(t)\\&-y(t-h(t))^T[3PG^TGP+4b_1P^2]y(t-h(t))\end{aligned} \tag{3.8}$$

where

$$\xi(t)^T=\left[y(t)^Ty(t-h(t))^Ty(t-h_1)^Ty(t-h_2)^T\int_{t-h_2}^{t-h_1}y(\theta)^Td\theta\quad\dot{y}(t)^T\right],$$

$$M=\begin{bmatrix}M_{11}&M_{12}&M_{13}&M_{14}&M_{15}&M_{16}\\ *&M_{22}&M_{23}&M_{24}&M_{25}&M_{26}\\ *&*&M_{33}&M_{34}&M_{35}&M_{36}\\ *&*&*&M_{44}&M_{45}&M_{46}\\ *&*&*&*&M_{55}&M_{56}\\ *&*&*&*&*&M_{66}\end{bmatrix},$$

$$M_{11}=AP+PA^T+2\beta P-R(e^{-2\beta h_1}+e^{-2\beta h_2})+\frac{4}{\gamma}CC^T+(2+4c_1+2c)Y^TY$$
$$+\varepsilon I-\frac{2e^{-4\beta h_2}(h_2-h_1)}{h_2+h_1}\Lambda+(BY+Y^TB^T)+2Q+3PE^TEP+(4a_1+2a)P^2,$$

$$M_{12}=DP,\quad M_{13}=e^{-2\beta h_1}R,\quad M_{14}=e^{-2\beta h_2}R$$
$$M_{15}=\frac{2e^{-4\beta h_2}(h_2-h_1)}{h_2+h_1}\Lambda,\quad M_{16}=PA^T+BY,$$
$$M_{22}=-2e^{-2\beta h_2}U+e^{-2\beta h_2}(S+S^T)+3PG^TGP+(4b_1+2b)P^2,$$
$$M_{23}=e^{-2\beta h_2}U-e^{-2\beta h_2}S,\quad M_{24}=e^{-2\beta h_2}U-e^{-2\beta h_2}S^T,\quad M_{25}=0,\quad M_{26}=PD^T$$
$$M_{33}=-e^{-2\beta h_1}Q-e^{-2\beta h_1}R-e^{-2\beta h_2}U$$
$$M_{34}=e^{-2\beta h_2}S^T,\quad M_{35}=0,\quad M_{36}=0$$
$$M_{44}=-e^{-2\beta h_2}Q-e^{-2\beta h_2}R-e^{-2\beta h_2}U,\quad M_{45}=0,\quad M_{46}=0$$
$$M_{55}=-\frac{2e^{-4\beta h_2}}{h_2^2-h_1^2}\Lambda,\quad M_{56}=0,$$
$$M_{66}=R(h_1^2+h_2^2)+(h_2-h_1)^2U+h_2(h_2-h_1)\Lambda-2P+\frac{4}{\gamma}CC^T+\varepsilon I$$

Using the Schur complement lemma, Proposition 2, the condition (3.2) is equivalent to the condition $M<0$ and from the inequality (3.8) it follows that

$$\begin{aligned}\dot{V}(\cdot)+2\beta V(\cdot)\le&\gamma\omega(t)^T\omega(t)-y(t)^T[3PE^TEP+4a_1P^2+(2+4c_1)Y^TY]y(t)\\&-y(t-h(t))^T[3PG^TGP+4b_1P^2]y(t-h(t))\end{aligned} \tag{3.9}$$

Letting $\omega(t)=0$, and since

$$-y(t)^T[3PE^TEP+4a_1P^2+(2+4c_1)Y^TY]y(t)\le 0,$$
$$-y(t-h(t))^T[3PG^TGP+4b_1P^2]y(t-h(t))\le 0,$$

we finally obtain from the inequality (3.9) that

$$\dot{V}(t,x_t)+2\beta V(t,x_t)\le 0. \tag{3.10}$$

Differentiating inequality (3.10) from 0 to t gives

$$V(t,x_t)\le V(0,x_0)e^{-2\beta t},\quad t\ge 0.$$

Taking the condition (3.3) into account, we obtain

$$\|x(t,\varphi)\|\le\sqrt{\frac{\alpha_2}{\alpha_1}}\|\varphi\|_{C_1}e^{-\beta t},\quad t\ge 0,$$

which implies that the zero solution of the closed-loop system is β-stable. To complete the proof of the theorem, it remains to show the γ-optimal level condition (ii). For this, we consider the following relation:

$$\int_0^s[\|z(t)\|^2-\gamma\|\omega(t)\|^2]\,dt=\int_0^s[\|z(t)\|^2-\gamma\|\omega(t)\|^2$$
$$+\dot{V}(t,x_t)]\,dt-\int_0^s\dot{V}(t,x_t)\,dt,\ \forall s\ge 0.$$

Since $V(t,x_t)\ge 0$, we have

$$-\int_0^s\dot{V}(t,x_t)\,dt=V(0,x_0)-V(s,x_s)\le V(0,x_0),\ \forall s\ge 0.$$

Therefore, for all $s\ge 0$

$$\int_0^s[\|z(t)\|^2-\gamma\|\omega(t)\|^2]\,dt\le\int_0^s[\|z(t)\|^2-\gamma\|\omega(t)\|^2+\dot{V}(t,x_t)]\,dt+V(0,x_0). \tag{3.11}$$

Combining the condition (3.3) and the inequality

$$V(t,x_t)\ge x(t)^TP_1x(t)=y(t)^TPy(t),$$

we obtain from (3.9) that

$$\begin{aligned}\dot{V}(t,x_t)\le&\gamma\omega(t)^T\omega(t)-y(t)^T[3PE^TEP+4a_1P^2+(2+4c_1)Y^TY]y(t)\\&-y(t-h(t))^T[3PG^TGP+4b_1P^2]y(t-h(t))-2\beta y(t)^TPy(t).\end{aligned} \tag{3.12}$$

Observe that the value of $\|z(t)\|^2$ is defined due to (2.1) and (3.1) as

$$\begin{aligned}\|z(t)\|^2\le&\|Ex(t)\|^2+\|Gx(t-h(t))\|^2+\|u(t)\|^2+2x(t)^TE^TGx(t-h(t))\\&+2x(t)^TE^Tg(\cdot)+2x(t-h(t))^TG^Tg(\cdot)+2u(t)^TF^Tg(\cdot)+\|g(\cdot)\|^2\\\le&3\|Ex(t)\|^2+3\|Gx(t-h(t))\|^2+2\|u(t)\|^2+4\|g(\cdot)\|^2\\\le&x(t)^T[3E^TE+4a_1]x(t)+x(t-h(t))^T[3G^TG+4b_1]x(t-h(t))\\&+[2+4c_1]\|u(t)\|^2\\=&y(t)^T[3PE^TEP+4a_1P^2+(2+4c_1)Y^TY]y(t)\\&+y(t-h(t))^T[3PG^TGP+4b_1P^2]y(t-h(t)).\end{aligned} \tag{3.13}$$

Submitting the estimation of $\dot{V}(t,x_t)$ and $\|z(t)\|^2$ defined by (3.12) and (3.13), respectively into (3.11), we obtain

$$\int_0^s[\|z(t)\|^2-\gamma\|\omega(t)\|^2]\,dt\le\int_0^s[-2\beta y(t)^TPy(t)]\,dt+V(0,x_0). \tag{3.14}$$

Hence, from (3.14) it follows that

$$\int_0^s[\|z(t)\|^2-\gamma\|\omega(t)\|^2]\,dt\le V(0,x_0)\le\alpha_2\|\varphi\|_{C_1}^2,$$

equivalently,

$$\int_0^s\|z(t)\|^2\,dt\le\int_0^s\gamma\|\omega(t)\|^2dt+\alpha_2\|\varphi\|_{C_1}^2,$$

Letting $s\to\infty$, and setting $c_0=\alpha_2/\gamma>0$, we obtain that

$$\frac{\int_0^\infty\|z(t)\|^2dt}{c_0\|\varphi\|_{C_1}^2+\int_0^\infty\|\omega(t)\|^2dt}\le\gamma,$$

for all non-zero $\omega(t)\in L_2([0,\infty],R^r)$, $\varphi(t)\in C^1([-\tau,0],R^n)$. This completes the proof of the theorem. □

In the sequel, we give an application to H_∞ control of uncertain linear systems with interval time-varying delay considered in [30,11,7]. Consider the following uncertain linear systems with time-varying delay:

$$\begin{cases}\dot{x}(t)=[A+\Delta A(t)]x(t)+[D+\Delta D(t)]x(t-h(t))\\+[B+\Delta B(t)]u(t)+[C+\Delta C]\omega(t),\quad t\in R^+,\\z(t)=[E+\Delta E]x(t)+[G+\Delta G]x(t-h(t))+[F+\Delta F]u(t),\end{cases} \tag{3.15}$$

where the time-varying uncertainties ΔA, ΔD, ΔB, ΔC, ΔE, ΔG, ΔF are given as

$$[\Delta A\ \Delta D\ \Delta B\ \Delta C\ \Delta E\ \Delta G\ \Delta F] = KH(t)[L_a\ L_d\ L_b\ L_c\ L_e\ L_g\ L_f],$$

K, $L_a\ L_d\ L_b\ L_c\ L_e\ L_g\ L_f$ are known real constant matrices of appropriate dimensions and $H(t)$ is an unknown uncertain matrix with Lebesgue measurable elements satisfying

$$H(t)^T H(t) \le I, \quad \forall t \ge 0. \tag{3.16}$$

To apply Theorem 1, let us denote

$$f(t,x,x^h,u,\omega) = \Delta Ax(t) + \Delta Dx(t-h(t)) + \Delta Bu(t) + \Delta C\omega(t),$$
$$g(t,x,x^h,u) = \Delta Ex(t) + \Delta Gx(t-h(t)) + \Delta Fu(t).$$
$$\lambda_K = \lambda_{max}(K^T K), \lambda_{L_a} = \lambda_{max}(L_a^T L_a),$$
$$\lambda_{L_b} = \lambda_{max}(L_b^T L_b), \lambda_{L_c} = \lambda_{max}(L_c^T L_c),$$
$$\lambda_{L_d} = \lambda_{max}(L_d^T L_d), \lambda_{L_e} = \lambda_{max}(L_e^T L_e),$$
$$\lambda_{L_g} = \lambda_{max}(L_g^T L_g), \lambda_{L_f} = \lambda_{max}(L_f^T L_f),$$

Observe that

$$\begin{aligned}\|\Delta Ax(t)\|^2 &= x(t)^T L_a^T H^T(t) K^T K H(t) L_a x(t)\\ &\le \lambda_K x(t)^T L_a^T H^T(t) H(t) L_a x(t) \le \lambda_K x(t)^T L_a^T L_a x(t)\\ &\le \lambda_K \lambda_{L_a} x(t)^T x(t),\end{aligned}$$
$$\|\Delta Dx(t-h(t))\|^2 \le \lambda_K \lambda_{L_d} \|x(t-h(t))\|^2,$$
$$\|\Delta Bu(t)\|^2 \le \lambda_K \lambda_{L_b} \|u(t)\|^2, \ \|\Delta C\omega(t)\|^2 \le \lambda_K \lambda_{L_c} \|\omega(t)\|^2$$
$$\|\Delta Ex(t)\|^2 \le \lambda_K \lambda_{L_e} \|x(t)\|^2,$$
$$\|\Delta Gx(t-h(t))\|^2 \le \lambda_K \lambda_{L_g} \|x(t-h(t))\|^2,$$
$$\|\Delta Fu(t)\|^2 \le \lambda_K \lambda_{L_f} \|u(t)\|^2,$$

and using the uncertain boundedness condition (3.16) we have

$$\begin{aligned}\|g(\cdot)\|^2 &\le 3\|\Delta Ex\|^2 + 3\|\Delta Gx^h\|^2 + 3\|\Delta Fu\|^2\\ &\le 3\lambda_K\lambda_{L_e}\|x\|^2 + 3\lambda_K\lambda_{L_g}\|x^h\|^2 + 3\lambda_K\lambda_{L_f}\|u\|^2.\end{aligned}$$

$$\begin{aligned}\|f(\cdot)\| &\le \|\Delta Ax\| + \|\Delta Dx^h\| + \|\Delta Bu\| + \|\Delta C\omega\|\\ &\le \sqrt{\lambda_K\lambda_{L_a}}\|x\| + \sqrt{\lambda_K\lambda_{L_d}}\|x^h\| + \sqrt{\lambda_K\lambda_{L_b}}\|u\| + \sqrt{\lambda_K\lambda_{L_c}}\|\omega\|\end{aligned}$$

By the same notations used in Theorem 1 and applying Theorem 1 with

$$a = \sqrt{\lambda_K\lambda_{L_a}}, \quad b = \sqrt{\lambda_K\lambda_{L_d}}, \quad c = \sqrt{\lambda_K\lambda_{L_b}}, \quad d = \sqrt{\lambda_K\lambda_{L_c}};$$
$$a_1 = 3\lambda_K\lambda_{L_e}, \quad b_1 = 3\lambda_K\lambda_{L_g}, \quad c_1 = 3\lambda_K\lambda_{L_f},$$

we have:

Corollary 1. *The H_∞ control of system (3.15) has a solution if there exist symmetric positive definite matrices P, Q, R, U, Λ and matrices Y, S such that the following LMI holds*:

$$\begin{bmatrix}\Omega_{11} & \Omega_{12}\\ * & \Omega_{22}\end{bmatrix} < 0. \tag{3.17}$$

Moreover, the stabilizing feedback control is given by

$$u(t) = YP^{-1}x(t), \quad t \ge 0,$$

and the solution of the system satisfies

$$\|x(t,\varphi)\Big| \le \sqrt{\frac{\alpha_2}{\alpha_1}}\|\varphi\|_{C_1} e^{-\beta t}, \quad t \ge 0.$$

Remark 1. Note that although the similar Lyapunov–Krasovskii functional was used in [25] to investigate the stability of systems with time-varying delay, the slack variables Λ_1 in Theorem 1 have not been introduced in [24] since in the derivation of stability only single and double integrals depending on the delay were used, while we use additional triple integral V_7. This Lyapunov–Krasovskii functional is mainly based on the information of the lower and upper delay bounds, which allows us to avoid using any assumption on the differentiability of the delay function. Therefore, our results are more comprehensive and effective. Theorem 1 provides sufficient conditions for the closed-loop system to be exponentially stable with a prescribed decay rate β, while the existing method can provide only asymptotic stability of the closed-loop system.

Remark 2. In the papers [31,27,30,11], additional unknowns and free weighting matrices are introduced to make the flexibility to solve the resulting LMIs. However, too many unknowns and free-weighting matrices employed in the existing methods complicate the system analysis and significantly increase the computational demand. Compared with the free matrix method used in [27,30,11], our simpler uncorrelated augmented matrix method uses fewer variables, e.g. LMI (3.2) has 7 unknown variables, meanwhile the LMI condition proposed in [27] has 12; in [30] has 24 and in [11] has 70. Moreover, in the previous papers the time delays are assumed to be differentiable and its derivative is bounded. In Theorem 1 this assumption is removed and LMI conditions (3.2) contain fewer unknown variables and then reduce computational complexity.

4. Numerical examples

In this section, we give numerical examples to show the validity of the H_∞ controller designed in the previous section.

Example 1. Consider the nonlinear system with interval time-varying delay (2.1), where

$$\begin{cases} h(t) = 0.1 + 0.4\sin^2(t) & \text{if } t \in H = \cup_{k\ge 0}(2k\pi, (2k+1)\pi),\\ h(t) = 0.1 & \text{if } t \notin H,\end{cases}$$

$$A = \begin{bmatrix}-1.3 & 0.3\\ -0.5 & 0.1\end{bmatrix}, \quad D = \begin{bmatrix}-0.01 & 0.02\\ 0.03 & -0.04\end{bmatrix},$$

$$B = \begin{bmatrix}0.2\\ 0.3\end{bmatrix}, \quad C = \begin{bmatrix}-0.02 & 0.01\\ 0.02 & -0.03\end{bmatrix},$$

$$E = G = \begin{bmatrix}0.06 & -0.06\\ -0.08 & 0.08\end{bmatrix}, \quad F = \begin{bmatrix}0.8\\ 0.6\end{bmatrix},$$

$$f(\cdot) = g(\cdot) = 0.01\begin{bmatrix}\sqrt{x_1^2(t) + x_2^2(t-h(t))}\\ \sqrt{x_2^2(t) + x_1^2(t-h(t))}\end{bmatrix}$$

and $a = b = c = d = a_1 = b_1 = c_1 = 0.01$.

It is worth noting that the delay function $h(t)$ is non-differentiable and the results obtained in [4,28,31,9] cannot be applicable to this system. By using LMI Toolbox in Matlab [5], the LMI (3.2) is feasible with $h_1 = 0.1$, $h_2 = 0.5$, $\beta = 0.1$, $\gamma = 4$, and

$$P = \begin{bmatrix}7.1005 & 3.4234\\ 3.4234 & 10.3682\end{bmatrix}, \quad Q = \begin{bmatrix}4.7944 & 1.2645\\ 1.2645 & 8.0441\end{bmatrix},$$

$$R = \begin{bmatrix}1.3189 & 1.2254\\ 1.2254 & 4.0863\end{bmatrix}, \quad U = \begin{bmatrix}9.2066 & 8.0385\\ 8.0385 & 23.2425\end{bmatrix},$$

$$\Lambda = \begin{bmatrix}30.4436 & 13.7492\\ 13.7492 & 70.8113\end{bmatrix},$$

$$Y = [-1\ \ -1], \quad S = \begin{bmatrix}-1 & 0.2\\ 0.6 & -1\end{bmatrix}.$$

The feedback control is given by

$$u(t) = YP^{-1}x(t) = [-0.1122\ \ -0.0594]x(t), \quad t \ge 0.$$

Moreover, the solution $x(t,\varphi)$ of the system satisfies

$$\|x(t,\varphi)\| \le 5.6402 e^{-0.1t}\|\varphi\|_{C_1}.$$

Fig. 1 shows the response solution $x(t)$ of the closed-loop system (2.1) with the initial condition $\varphi(t) = [-4\ 4]^T$.

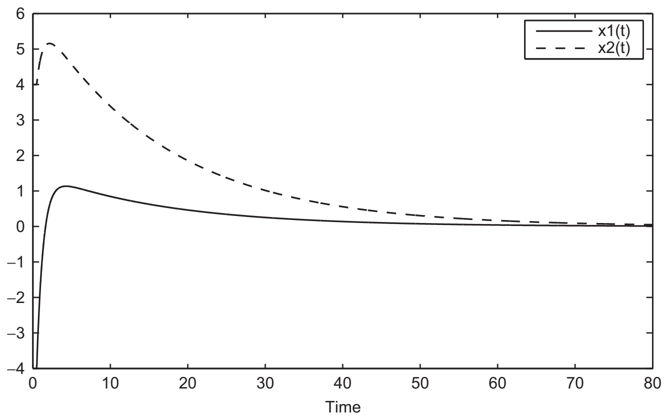

Fig. 1. Response solution of the system (2.1).

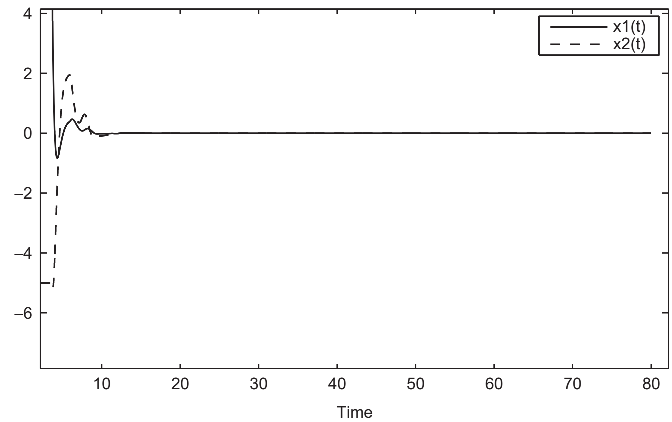

Fig. 2. Response solution of the system (3.15).

Example 2. Consider the uncertain linear systems with interval time-varying delay (3.15), where

$$\begin{cases} h(t)=2+1.7\sin^2(t) & \text{if } t\in H=\underset{k\geq 0}{\cup}(2k\pi,(2k+1)\pi), \\ h(t)=2 & \text{if } t\notin H, \end{cases}$$

$$A=\begin{bmatrix}-3 & 2\\ -1 & 0.1\end{bmatrix},\quad D=\begin{bmatrix}-0.1 & 0.1\\ 0.3 & -0.3\end{bmatrix},$$

$$B=\begin{bmatrix}3\\5\end{bmatrix},\quad C=\begin{bmatrix}-0.02 & 0.01\\ 0.02 & -0.03\end{bmatrix},$$

$$E=G=\begin{bmatrix}0.06 & -0.06\\ -0.08 & 0.08\end{bmatrix},\quad F=\begin{bmatrix}0.8\\0.6\end{bmatrix},$$

$$K=L_a=L_d=L_b=L_c=\begin{bmatrix}0.1 & 0\\ 0 & 0.1\end{bmatrix},\quad L_e=L_g=L_f=\begin{bmatrix}0.03 & 0\\ 0 & 0.03\end{bmatrix}$$

It is worth noting that, the delay function $h(t)$ is non-differentiable and the results obtained in [30,11,7] are not applicable to this system. By using LMI Toolbox in Matlab, the LMI (3.17) is feasible with $h_1=2$, $h_2=3.7$, $\beta=0.1$, $\gamma=4$, and

$$P=\begin{bmatrix}2.8078 & 2.7685\\ 2.7685 & 3.2946\end{bmatrix},\quad Q=\begin{bmatrix}0.0276 & -0.0622\\ -0.0622 & 0.1495\end{bmatrix},$$

$$R=10^{-3}\begin{bmatrix}0.1641 & 0.1822\\ 0.1822 & 0.2033\end{bmatrix},\quad U=\begin{bmatrix}0.9909 & 1.1102\\ 1.1102 & 1.3080\end{bmatrix},$$

$$\Lambda=10^{-3}\begin{bmatrix}0.4483 & 0.4936\\ 0.4936 & 0.5465\end{bmatrix},$$

$$Y=[-0.3259\ \ -0.2143],\quad S=\begin{bmatrix}0.4263 & 0.4970\\ 0.4994 & 0.5174\end{bmatrix}.$$

The feedback control is given by

$$u(t)=YP^{-1}x(t)=[-0.3029\ \ 0.1895]x(t),\quad t\geq 0.$$

Moreover, the solution $x(t,\varphi)$ of the system satisfies

$$\|x(t,\varphi)\|\leq 29943e^{-0.1t}\|\varphi\|_{C_1}.$$

Fig. 2 shows the response solution $x(t)$ of the closed-loop system of system (3.15) with the initial condition $\varphi(t)=[5\ -5]^T$.

5. Conclusion

In this paper, the problem of H_∞ control for nonlinear systems with interval time-varying delays has been studied. By introducing a set of improved Lyapunov–Krasovskii functionals and using new bounding estimation technique, delay-dependent conditions for the H_∞ control and exponential stability have been established in terms of LMIs. An application to H_∞ control of uncertain linear systems with interval time-varying delay has been given. Numerical examples are given showing the effectiveness of the obtained results.

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