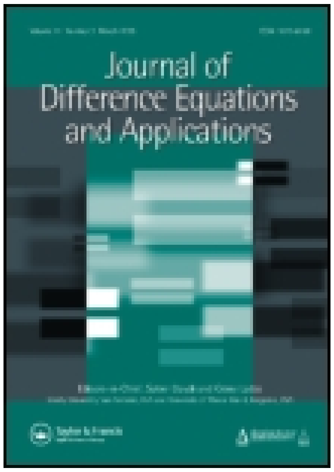

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Asymptotically almost periodic solutions on the half-line

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Asymptotically almost periodic solutions on the half-line

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In this paper, we first show that if x is a bounded solution of the difference equation $x(n+1) = Ax(n) + f(n), \forall n \in \mathbb{N}$ and the sequence $\{x(n)\}_{n\in\mathbb{N}}$ is totally ergodic, $\sigma_\Gamma(A) := \sigma(A) \cap \Gamma$ is countable and the sequence $\{f(n)\}_{n\in\mathbb{N}}$ is asymptotically almost periodic, then the sequence $\{x(n)\}_{n\in\mathbb{N}}$ is asymptotically almost periodic. As an application, we consider the asymptotical periodicity of mild solutions to periodic evolution equations.

Keywords: Complex Banch space; Periodic evolution; Spectral theory; Discretized equation

1. Introduction

Let us consider the difference equation

$$x(n+1) = Ax(n) + f(n), \quad \forall n \in \mathbb{N},$$

where A is a bounded linear operator on a Banach space $\mathbb{X}$, the sequence $\{f(n)\}_{n\in\mathbb{N}}$ is asymptotically almost periodic. We will study a spectral criteria for the existence of asymptotically almost periodic solutions for the above equation and apply them to study a similar problem for evolution equation of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = A(t)x + f(t), \quad t \in \mathbb{R}^+,$$

where $A(t)$ is a (in general unbounded) linear operator on $\mathbb{X}$, which is periodic and f is asymptotically almost periodic.

One of the central topics in the qualitative theory of difference equations and differential equations is to find conditions for the existence of asymptotically almost periodic solutions. As is well-known, there is a close relation between the behavior of a differential equation and its discretized equation. This is a motivation of many works on the asymtotic behavior of difference equations. In this paper, we will first consider the asymptotic periodicity of solutions to a difference equation by means of spectral theory of sequences. As a result, we obtain discrete analogs of several results in [2] that will be then applied to periodic evolution equations to show that a bounded mild solution x is asymptotically almost periodic if f is asymptotically almost periodic, the sequence $\{x(n)\}_{n\in\mathbb{N}}$ is bounded and totally ergodic

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and the spectrum of the monodromy operator of the equation contains only countably many points on the unit cirle. Note that in [3], apart from the coutability conditions of the part of spectrum of the monodromy operator on the unit cirle, the authors obtained different conditions in terms of the boundedness, uniform continuity and total ergodicity of $x(\cdot)$. The main results of this paper are stated in Theorems 3.1, 4.3.

2. Preliminaries

The section will be devoted entirely to the notation and concept of almost periodic sequence on the line, on the half-line. Almost all results of this section are more or less known. However, for the reader's convenience we will quote them here and even verify several results which seem to be obvious but not available in the mathematical literature.

Throughout this paper we will use the following notations: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{C}$ stand for the sets of natural, integer, real, non negative real, complex numbers, recpectively. X denotes a given complex Banach space. As usual, $\sigma(A)$, $\rho(A)$, $R(\lambda, A)$ are notations of the spectrum, resolvent set and resolvent of an operator A. The notations $BC(\mathbb{R}, \mathbb{X})$, $BUC(\mathbb{R}, \mathbb{X})$, $AP(\mathbb{R}, \mathbb{X})$ will stand for the spaces of all $\mathbb{X}$-valued bounded, bounded uniformly continuous on $\mathbb{R}$ and its subspace of almost periodic (in Bohr's sense) functions, respectively. Recall that a function $h \in BUC(\mathbb{R}, \mathbb{X})$ is called almost periodic (in Bohr's sense) if the set $\{S(t)h, t \in \mathbb{R}\}$ is relatively compact in $BUC(\mathbb{R}, \mathbb{X})$, where $(S(t))_{t\in\mathbb{R}}$ is the group of translations on $BUC(\mathbb{R}, \mathbb{X})$. The notations $BC(\mathbb{R}^+, \mathbb{X})$, $BUC(\mathbb{R}^+, \mathbb{X})$, $AP(\mathbb{R}^+, \mathbb{X})$ will stand for the spaces of all $\mathbb{X}$-valued bounded, bounded uniformly continuous on $\mathbb{R}^+$ and its subspace of almost periodic (in Bohr's sense) functions, respectively. In this case, a function $h \in BUC(\mathbb{R}^+, \mathbb{X})$ is called almost periodic (in Bohr's sense) if the set $\{S(t)h, t \in \mathbb{R}^+\}$ is relatively compact in $BUC(\mathbb{R}^+, \mathbb{X})$, where $(S(t))_{t\in\mathbb{R}^+}$ is the semigroup of translations on $BUC(\mathbb{R}^+, \mathbb{X})$. We denote by $l_\infty(\mathbb{X}), l_\infty^+(\mathbb{X})$ the spaces of all two-side, one side sequences with sup-norm, respectively, i.e.

$$l_\infty(\mathbb{X}) := \Big\{ \{x(n)\}_{n\in\mathbb{Z}} : x(n) \in \mathbb{X}, \sup_{n\in Z} \|x(n)\| < +\infty \Big\},$$

$$l_\infty^+(\mathbb{X}) := \Big\{ \{x(n)\}_{n\in\mathbb{N}} : x(n) \in \mathbb{X}, \sup_{n\in\mathbb{N}} \|x(n)\| < +\infty \Big\}.$$

The notation c_0 will stand for the subspace of $l_\infty^+(\mathbb{X})$ containing all sequences which converge to 0. The group of translations $(S(n))_{n\in\mathbb{Z}}$ on $l_\infty(\mathbb{X})$ is defined as follows:

$$S(k)x := \{x(n+k)\}_{n\in Z}, \quad x := \{x(n)\}_{k\in Z}, \quad \forall k \in \mathbb{Z}.$$

The subspace of $l_\infty(\mathbb{X})$ consisting of all almost periodic sequences is denoted by $AP(\mathbb{Z},\mathbb{X})$. In this case, a sequence $x \in l_\infty(\mathbb{X})$ is called almost periodic if the set $\{S(n)x, n \in \mathbb{Z}\}$ is relatively compact in $l_\infty(\mathbb{X})$.

The semigroup of translations $(S(n))_{n\in\mathbb{N}}$ on $l_\infty^+(\mathbb{X})$ is defined as follows:

$$S(k)x := \{x(n+k)\}_{n\in\mathbb{N}}, \quad x := \{x(n)\}_{n\in\mathbb{N}}, \quad \forall k \in \mathbb{N}.$$

The subspace of $l_\infty^+(\mathbb{X})$ consisting of all almost periodic sequences is denoted by $AP(\mathbb{N},\mathbb{X})$, i.e.

$$AP(\mathbb{N},\mathbb{X}) := \overline{\mathrm{span}\{\{\lambda^n y\}_{n\in\mathbb{N}}, \quad y\in\mathbb{X}, \quad \lambda\in\Gamma\}},$$

where $\Gamma := \{z\in\mathbb{C} : |z| = 1\}$. For more details, we refer the reader to [1,4,6].

2.1 Almost periodic sequences

In this section, we will show some equivalent definitions of almost periodic sequence in $l_\infty(\mathbb{X})$, which follows from the Theorem 2.1, 2.2.

A subset Q of $\mathbb{J}$ is called relatively dense in $\mathbb{J}$ if there exists a length $l > 0$ such that

$$Q\cap[a, a+l] \neq \emptyset, \quad \text{for all} \;\; a\in\mathbb{J},$$

where $\mathbb{J}$ is one of the four set: $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{Z}$, $\mathbb{N}$.

THEOREM 2.1 *Let $x\in l_\infty(\mathbb{X})$. Then the following assertions are equivalent.*

(i) *The set $\{S(n)x, n\in\mathbb{Z}\}$ is relatively compact in $l_\infty(\mathbb{X})$.*
(ii) *The set $\{S(n)x, n\in\mathbb{N}\}$ is relatively compact in $l_\infty(\mathbb{X})$.*
(iii) *For all $\varepsilon > 0$, the set $Q_{\varepsilon,x} := \{\tau\in\mathbb{Z} : \|S(\tau)x - x\| \leq \varepsilon\}$ is relatively dense in $\mathbb{Z}$.*

Proof.

(i) (i) ⇒ (ii). This is trivial.
(ii) (ii) ⇒ (iii). Let $\varepsilon > 0$. By assumption, there exists $n_1, n_2, \ldots, n_m \in\mathbb{N}$ such that for each $n\in\mathbb{N}$, there exists $j\in\{1,2,\ldots,m\}$ such that $\|S(n)x - S(n_j)x\| \leq \varepsilon$. Let $l := \max_{j\in\overline{1,m}} n_j$. We will show that $Q_{\varepsilon,x}\cap[n, n+l] \neq \emptyset, \quad \forall n\in\mathbb{Z}$.

Let $n \geq 0$. Choose $j\in\{1,2,\ldots,m\}$ such that $\|S(n+l)x - S(n_j)x\| \leq \varepsilon$. Let $\tau := n + l - n_j$. Then,

$$\|S(\tau)x - x\| = \|S(-n_j)(S(n+l)x - S(n_j)x\| = \|S(n+l)x - S(n_j)x\| \leq \varepsilon.$$

Thus $\tau\in Q_{\varepsilon,x}\cap[n,\, n+l]$. It leads to $Q_{\varepsilon,x}\cap[n, n+l] \neq \emptyset$ for all $n \geq 0$.

Let $n < 0$. Then there exists $j\in\{1,2,\ldots,m\}$ such that $\|S(-n)x - S(n_j)x\| \leq \varepsilon$. Let $\tau := n + n_j$. Then

$$\|S(\tau)x - x\| = \|S(n)(S(-n)x - S(n_j)x\| = \|S(-n)x - S(n_j)x\| \leq \varepsilon.$$

Thus $\tau\in Q_{\varepsilon,x}\cap[n,\, n+l]$ for $n < 0$.

Hence, $Q_{\varepsilon,x}\cap[n, n+l] \neq \emptyset$ for all $n\in\mathbb{Z}$.
(iii) (iii) ⇒ (i). Let $\varepsilon > 0$. By assumption, there exists $l\in\mathbb{N}$ such that for all $n\in\mathbb{Z}$, $Q_{\varepsilon,x}\cap[\mathrm{l}n, \mathrm{l}n + l] \neq \emptyset$. Let $t\in\mathbb{Z}$. Take $n\in\mathbb{Z}$ such that $t\in[\mathrm{ln},\, \mathrm{ln} + l]$ and choose $\tau\in Q_{\varepsilon,x}\cap[-\mathrm{ln}, -\mathrm{ln} + l]$. Then $(t+\tau)\in[0,2l]$. There exists $j\in\{1,2,\ldots,2l\}$ such

that $t+\tau=j$. Thus

$$\|S(t)x - S(j)x\| = \|S(t)x - S(t)S(\tau)x\| = \|x - S(\tau)x\| \leq \varepsilon.$$

Hence, the orbit $\{S(t)x, t\in\mathbb{Z}\}$ is covered by balls $B_\varepsilon(S(j)x) := \{z \in l_\infty(\mathbb{X}) : \|z - S(j)x\| < \varepsilon\}(j \in \{1, 2, \ldots, 2l\})$. Thus (i) is proved.

□

THEOREM 2.2 *Let $x \in l_\infty(\mathbb{X})$. Then the following assertions are equivalent.*

(i) $x \in \overline{\operatorname{span}\{\{\lambda^n y\}_{n\in\mathbb{Z}}, \lambda \in \Gamma, y \in \mathbb{X}\}}$.
(ii) *The set $Q_{\varepsilon,x} := \{\tau \in \mathbb{Z} : \|S(\tau)x - x\| \leq \varepsilon\}$ is relatively dense in $\mathbb{Z}$ for all $\varepsilon \geq 0$.*
(iii) $x \in AP(\mathbb{Z}, \mathbb{X})$.

In order to prove the theorem, we require the following lemmas.

LEMMA 2.3 *Let x be a sequence in $l_\infty(\mathbb{X})$ defined by the following formula*

$$x(n) = \sum_{k=1}^{m} y_k \lambda_k^n, \quad n \in \mathbb{Z}, \quad y_k \in \mathbb{X}, \quad \lambda_k \in \Gamma, \quad \forall k = \overline{1, m}.$$

Then, the sequence x is almost periodic.

Proof. We define a function $h : \mathbb{R} \to \mathbb{X}$ by the following formula

$$h(t) = \sum_{k=1}^{m} y_k \lambda_k^t.$$

It is well-known that the function h is almost periodic, this is equivalent to the set $\{S(t)h, t \in \mathbb{R}\}$ relatively compact in $BUC(\mathbb{R},\mathbb{X})$. Thus, for any sequences $\{S(n_k)h, n_k \in \mathbb{Z}\}$, we can extract a subsequence, which is denoted again by $\{S(n_k)h, n_k \in \mathbb{Z}\}$, converges to $h_0 \in BUC(\mathbb{R}, \mathbb{X})$, i.e.

$$\lim_{k\to+\infty} \sup_{r\in\mathbb{R}} \|S(n_k)h(r) - h_0(r)\| = 0;$$

Consequently

$$\lim_{k\to+\infty} \sup_{r\in\mathbb{Z}} \|S(n_k)h(r) - h_0(r)\| = \lim_{k\to+\infty} \sup_{r\in\mathbb{Z}} \|S(n_k)x(r) - h_0(r)\| = 0.$$

Hence, $\{S(n)x, n \in \mathbb{Z}\}$ is relatively compact in $l_\infty(\mathbb{X})$. Applying the well-known result of Theorem 2.1, the sequence x should be almost periodic.

□

LEMMA 2.4 *Let a sequence $\{x^n\}_{n\in\mathbb{Z}} \subset AP(\mathbb{Z}, \mathbb{X})$, which converges to $x \in l_\infty(\mathbb{X})$ with sup-norm. Then, $x \in AP(\mathbb{Z}, \mathbb{X})$.*

Proof. Let a given $\varepsilon > 0$ and $a \in \mathbb{Z}$. Take n is large enough such that $\|x^n - x\| \leq (\varepsilon/3)$ and $\tau \in Q_{(\varepsilon/3),x^n} \cap \left[a, a + l\big((\varepsilon/3), x^n\big)\right]$. From

$$\|S(\tau)x - x\| \leq \|S(\tau)x - S(\tau)x^n\| + \|S(\tau)x^n - x^n\| + \|x - x^n\| \leq 3\frac{\varepsilon}{3} = \varepsilon$$

we have $\tau \in Q_{\varepsilon,x} \cap \left[a, a + l\big((\varepsilon/3), x^n\big)\right]$. This completes the proof of the lemma.

□

PROPOSITION 2.5 *Let a sequence $x \in AP(\mathbb{Z}, \mathbb{X})$. Then, the function $h : \mathbb{R} \to \mathbb{X}$ defined by the following formula*

$$h(t) := sx(n) + (1 - s)x(n + 1),\ \forall t \in \mathbb{R} : t = sn + (1 - s)(n + 1),\ n \in \mathbb{Z},\ s \in [0, 1],$$

is almost periodic.

Proof of Theorem 2.2. This theorem is implied from Theorem 2.1, Proposition 2.5 and the two above lemmas.

□

COROLLARY 2.6 *Let $x \in AP(\mathbb{Z}, \mathbb{X})$. Then there exists a sequence $\{t_n\} \subset \mathbb{Z}$, $t_n \to +\infty$ such that $\|S(t_n)x - x\| \leq (1/n)$. Moreover, for every $\tau \in \mathbb{Z}$ we have*

$$\|x\|_\infty = \sup_{t \geq \tau} \|x(t)\|.$$

2.2 Asymptotically almost periodic sequences on the half-line

The main result of the subsection is stated in Theorem 2.14, which says that a totally ergodic bounded sequence with countable spectrum is asymptotically almost periodic.

LEMMA 2.7 *For each sequence $x \in AP(\mathbb{N}, \mathbb{X})$, there exists a unique sequence $x_e \in AP(\mathbb{Z}, \mathbb{X})$ such that*

(i) $x_e(n) = x(n)$, $\forall n \in \mathbb{N}$,
(ii) $\|x_e\| = \|x\| = \sup_{t \geq \tau} \|x(t)\|$, $\forall \tau \in \mathbb{N}$.

Proof. By definition, there exists a sequence of trigonometric polynomials

$$x^n(t) = \sum_{k=1}^{m_n} y_{n_k} \lambda_{n_k}^t,\ y_{n_k} \in \mathbb{X},\ \lambda_{n_k} \in \Gamma,\ t \in \mathbb{N},$$

such that $\lim_{n\to+\infty} \|x^n - x\| = 0$. We define a sequence of trigonometric polynomials $x_e^n \in l_\infty(\mathbb{X})$ by expanding x^n onto $\mathbb{Z}$. Since $\lim_{n,m\to\infty} \sup_{s\in\mathbb{N}} \|x^n(s) - x^m(s)\| = 0$, it follows from corollary 2.6 that $x^n_{e\,n\in\mathbb{N}}$ is a Cauchy sequence of $l_\infty(\mathbb{X})$. Hence, there exists $x_e \in AP(\mathbb{Z}, \mathbb{X})$ which is the limit of the sequence. Obviously, x_e satisfies (i) and (ii).

Suppose that, there exists another sequence $\{x'_e\}$ satisfying the two properties. Then,

$$\|x'_e - x_e\| = \sup_{n\geq 0} \|x'_e(n) - x_e(n)\| = \sup_{n\geq 0} \|x(n) - x(n)\| = 0,$$

(see corollary 2.6). Hence, $x'_e = x_e$.

□

By using the lemma 2.7, it is easy to see

$$AP(\mathbb{N}, \mathbb{X}) \cap c_0 = \{0\}.$$

By

$$AAP(\mathbb{N}, \mathbb{X}) := c_0 \oplus AP(\mathbb{N}, \mathbb{X})$$

we denote the space of all asymptotically almost periodic sequences on the half-line. For $x = x_0 + x_1$ with $x_0 \in c_0, x_1 \in AP(\mathbb{N}, \mathbb{X})$, obviously $\|x\| \geq \|x_1\|$. Thus $AAP(\mathbb{N}, \mathbb{X})$ is a closed subspace of $l^+_\infty(\mathbb{X})$.

For $x \in l^+_\infty(\mathbb{X})$ and $\lambda_0 \in \Gamma$, we say that x is *uniformly ergodic* at λ_0 if there exists the limit

$$M_{\lambda_0}(x)(k) = \lim_{\alpha\downarrow 0} \alpha\hat{x}_k(\lambda_0 \exp(\alpha)), \; \forall k \in \mathbb{N},$$

where $x_k = S(k)x$ and $\hat{x}(\lambda) = \sum_{n=0}^{+\infty} \lambda^{-n-1} S(n)x$ for all $\lambda \in \mathbb{C}, |\lambda| > 1$. This is equivalent to the convergence of $M_{\lambda_0}(x) = \lim_{\alpha\downarrow 0} \alpha\hat{x}(\lambda_0 \exp(\alpha))$.

A sequence $x \in l^+_\infty(\mathbb{X})$ is called *totally ergodic* if x is uniformly ergodic at each $\lambda_0 \in \Gamma$.

A simple calculation shows that if x is uniformly ergodic at λ_0 then there exists $y_{\lambda_0,x} \in \mathbb{X}$ such that $M_{\lambda_0}(x) = \{\lambda_0^n . y_{\lambda_0,x}\}_{n\in\mathbb{N}}$.

Consequently, $M_{\lambda_0}(x)$ belongs to the space $AP(N;X)$ whenever x is uniformly ergodic at λ_0.

Now, we will consider the quotient space

$$Y := l^+_\infty(\mathbb{X})/AAP(\mathbb{N}, \mathbb{X}),$$

with quotient map $\pi : l^+_\infty(\mathbb{X}) \to Y$. Then Y is a Banach space with the norm

$$\|\pi(x)\|_Y := \inf_{g\in AAP(\mathbb{N},\mathbb{X})} \{\|x - g\|\}, \; \forall x \in l^+_\infty(\mathbb{X}).$$

We denote $\pi(x)$ by $\bar{x}$. Since $l^+_\infty(\mathbb{X})$, $AAP(\mathbb{N}, \mathbb{X})$, $AP(\mathbb{N}, \mathbb{X})$ are invariant under the shift semigroup, we can define a semigroup $\overline{(S(n))}_{n\in\mathbb{N}}$ on Y by

$$\overline{S(n)}\bar{x} = \overline{(S(n)x)}.$$

The interesting fact about this construction is the following:

Lemma 2.8 *The operator $\overline{S}$ is isometric and surjective, where $\overline{S} := \overline{S(1)}$.*

Proof. We first show that

$$\|\bar{S}\bar{x}\| = \|\bar{x}\|, \quad \forall x \in l^+_\infty(\mathbb{X}).$$

In fact,

$$\|\bar{S}\bar{x}\| = \inf_{g\in AAP(N,X)} \|Sx + g\| \leq \inf_{g\in AAP(N,X)} \|Sx + Sg\| \leq \inf_{g\in AAP(N,X)} \|x + g\| = \|\bar{x}\|.$$

Hence,

$$\|\bar{S}\bar{x}\| \leq \|\bar{x}\|. \tag{1}$$

For $x \in l_\infty^+(\mathbb{X})$ and $g \in AAP(\mathbb{N}, \mathbb{X})$, we define a sequence $h : \mathbb{N} \to \mathbb{X}$ by the formula

$$h(n) := \begin{cases} g(n-1), & n \geq 1, \\ x(0), & n = 0. \end{cases}$$

Then $h \in AAP(\mathbb{N}, \mathbb{X})$ (see proposition 2.7) and

$$\|x - h\| = \sup_{n \geq 1} \|x(n) - g(n-1)\| = \|Sx - g\|.$$

Thus,

$$\|\bar{x}\| \leq \|x - h\| = \|Sx - g\|.$$

Since g is arbitrary so that

$$\|\bar{x}\| \leq \inf_{g \in AAP(\mathbb{N}, \mathbb{X})} \|Sx - g\| = \|\bar{S}\bar{x}\|. \tag{2}$$

It follows from equations (1) and (2) that

$$\|\bar{S}\bar{x}\| = \|\bar{x}\|, \quad \forall x \in l_\infty^+(\mathbb{X}).$$

Now, we prove the surjectivity of $\bar{S}$. For arbitrary $x \in l_\infty^+(\mathbb{X})$, we define a sequence $g \in l_\infty^+(\mathbb{X})$ as follows,

$$g(n) := \begin{cases} x(n-1), & n \geq 1, \\ 0, & n = 0. \end{cases}$$

It is obvious that

$$\bar{S}\bar{g} = \bar{x}.$$

This proves the proposition.

□

We will denote by $\overline{M_x}$ for $x \in l_\infty^+(\mathbb{X})$ the closure of the subspace of Y spanned by all elements $\overline{S(n)x}$, $n \in \mathbb{Z}$. In this case, we define

$$S(-n)x = (\underbrace{0, 0, \ldots, 0}_{n \text{ elements}}, x(0), x(1), \ldots), n \in \mathbb{N}.$$

Lemma 2.9 *Let $x \in l_\infty^+(\mathbb{X})$. Then $\overline{M_x}$ is invariant under $\bar{S}$ and $\bar{S}_{|\overline{M_x}}$ is isometric, surjective from $\overline{M_x}$ onto $\overline{M_x}$.*

Proof. It is clear from the definition of $\overline{S(n)}$ that $\bar{S}\overline{(S(n)x)} \in \overline{M_x}$ for all $n \in \mathbb{Z}$. Let $\bar{g} \in \overline{M_x}$, then there exists a sequence $\{\overline{g^n}\}_{n \in \mathbb{N}} \subset \overline{M_x}$ defined as follows

$$\overline{g^n} = \sum_{k=1}^{m_n} \alpha_{n_k} \overline{S(n_k)x}, \quad n_k \in \mathbb{Z}, \ \alpha_{n_k} \in \mathbb{C},$$

which converges to $\bar{g}$. From

$$\lim_{n\to\infty}\|\bar{S}\bar{g}-\bar{S}\overline{g^n}\| = \lim_{n\to\infty}\|\bar{S}(\bar{g}-\overline{g^n})\| = \lim_{n\to\infty}\|\bar{g}-\overline{g^n}\| = 0,$$

we have $\bar{S}\bar{g}\in\overline{M_x}$, i.e. $\overline{M_x}$ is invariant under $\bar{S}$. We will show that $\bar{S}_{|\overline{M_x}}$ is surjective. We see that for all $k\in\mathbb{Z}$ then $\overline{S(k)x}=\bar{S}\overline{(S(k-1)x)}$, it leads to

$$\overline{g^n}=\sum_{k=1}^{m_n}\alpha_{n_k}\overline{S(n_k)x}=\bar{S}\left(\sum_{k=1}^{m_n}\alpha_{n_k}\overline{S(n_k-1)x}\right)=\overline{S(h^n)},$$

and $\|\overline{g^m}-\overline{g^n}\|=\|h^m-h^n\|$, where $h^n=\sum_{k=1}^{m_n}\alpha_{n_k}\overline{S(n_k-1)x}$, $\forall n,m\in\mathbb{N}$. Since $\{\overline{g^n}\}_{n\in\mathbb{N}}$ is a Cauchy sequence in the Banach space $\overline{M_x}$, it follows that $\{h^n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in the Banach space $\overline{M_x}$ and the limit belongs to $\overline{M_x}$, i.e. $\bar{S}_{|\overline{M_x}}$ is surjective.

□

Remark 2.10 It follows from Lemma 2.9 that $\sigma(\bar{S}_{|\overline{M_x}})\subset\Gamma$.

DEFINITION 2.11 *Let $x\in l_\infty^+(\mathbb{X})$. The subset of all $\lambda_0\in\Gamma$ at which*

$$\widehat{x_{AP}}(\lambda):=\sum_{n=0}^{+\infty}\lambda^{-n-1}\overline{S(n)}\bar{x},\quad \forall\lambda\in\mathbb{C},|\lambda|>1,$$

has no holomorphic extension to any neighborhood of λ_0, is said to be the quotient spectrum of the sequence $x\in l_\infty^+(\mathbb{X})$ and will be denoted by $\sigma_{AP}^+(x)$.

LEMMA 2.12 *Let $x\in l_\infty^+(\mathbb{X})$ and $\lambda_0\in\Gamma$, $0<\varepsilon<(1/2)$. Suppose that $\widehat{x_{AP}}$ has a holomorphic extension to $B_\varepsilon(\lambda_0)$. Then for all $\bar{g}\in\overline{M_x}$, $\widehat{g_{AP}}$ has a holomorphic extension to this neighborhood.*

Proof. Let $\bar{g}\in\overline{M_x}$, there exists a sequence $\{\overline{g^n}\}_{n\in\mathbb{N}}$ defined in a similar way to the sequence of the lemma 2.9, which converges to $\bar{g}$ and $\|\overline{g^n}-\bar{g}\|\le 1$ for all $n\in\mathbb{N}$.

It is obvious that $\widehat{g^n_{AP}}$ has a holomorphic extension to this neighborhood for all $n\in\mathbb{N}$ and $\widehat{g^n_{AP}}$ uniformly converges to $\widehat{g_{AP}}$ on every compact subset of the set $\{\lambda\in\mathbb{C},|\lambda|>1\}$. Let a given $\mu\in\mathbb{C}$, $|\mu|>1$. Then $k_n(\lambda)=(\lambda-\bar{S}_{|\overline{M_x}})R(\mu;\bar{S}_{|\overline{M_x}})\widehat{g^n_{AP}}(\lambda)$ defines a holomorphic function on $B_\varepsilon(\lambda_0)$ and $k_n(\lambda)=R(\mu;\bar{S}_{|\overline{M_x}})\overline{g^n}$ for $n\in\mathbb{N}$, $\lambda\in B_\varepsilon(\lambda_0)$, $|\lambda|>1$. By the uniqueness theorem, $k_n(\lambda)=R(\mu;\bar{S}_{|\overline{M_x}})\overline{g^n}$ for $n\in\mathbb{N}$, $\lambda\in B_\varepsilon(\lambda_0)$. Hence,

$$\left(\lambda-\bar{S}_{|\overline{M_x}}\right)\widehat{g^n_{AP}}(\lambda)=\overline{g^n},\quad\forall\lambda\in B_\varepsilon(\lambda_0),\quad n\in\mathbb{N}.$$

From isometricity of the operator $\bar{S}_{|\overline{M_x}}$ we have

$$\|\widehat{g^n_{AP}}(\lambda)\|\le\frac{\|\bar{g}\|+1}{|1-|\lambda||},\ \forall\lambda\in B_\varepsilon(\lambda_0)\setminus\Gamma.$$

Using exactly the argument of the proof of the proposition 2.3, page 4 in [5], we can show that $\widehat{g_{AP}}(\lambda)$ has a holomorphic extension onto $B_\varepsilon(\lambda_0)$.

□

LEMMA 2.13 *Let $x \in l_\infty^+(\mathbb{X})$. Then $\sigma(\bar{S}_{|\overline{M_x}}) \subset \sigma_{AP}^+(x)$.*

Proof. Let $\lambda_0 \in \Gamma$, $\lambda_0 \notin \sigma_{AP}^+(x)$. Then there exists a neighbourhood $B_\varepsilon(\lambda_0)$, $0 < \varepsilon < (1/2)$ to which $\widehat{x_{AP}}$ has a holomorphic extension.

Since,

$$\widehat{g_{AP}}(\lambda) = R\Big(\lambda, \bar{S}_{|\overline{M_x}}\Big)\bar{g}, \quad \forall |\lambda| > 1, \bar{g} \in \overline{M_x},$$

we have

$$\Big\|R\Big(\lambda; \bar{S}_{|\overline{M_x}}\Big)\bar{g}\Big\| = \|\widehat{g_{AP}}(\lambda)\| \leq \sup_{\lambda \in \overline{B_{(\varepsilon/2)}(\lambda_0)}, |\lambda|>1} \|\widehat{g_{AP}}(\lambda)\| < +\infty,$$

for all $\lambda \in \overline{B_{(\varepsilon/2)}(\lambda_0)}$, $|\lambda| > 1$. It follows from uniformly bounded principle that

$$\sup_{\lambda \in \overline{B_{(\varepsilon/2)}(\lambda_0)}, |\lambda|>1} \Big\|R\Big(\lambda; \bar{S}_{|\overline{M_x}}\Big)\Big\| < +\infty.$$

Hence, $\lambda_0 \notin \sigma\Big(\bar{S}_{|\overline{M_x}}\Big)$. □

THEOREM 2.14 *Let $x \in l_\infty^+(\mathbb{X})$ be totally ergodic and $\sigma_{AP}^+(x)$ is countable. Then $x \in AAP(\mathbb{N}, \mathbb{X})$.*

Proof. We will prove the theorem by contradiction. Suppose that $x \notin AAP(\mathbb{N}, \mathbb{X})$. Then, $\overline{M_x}$ is a non trivial Banach space and $\sigma(\bar{S}_{|\overline{M_x}})$ is non empty.

Since $\sigma(\bar{S}_{|\overline{M_x}}) \subset \sigma_{AP}^+(x)$ and $\sigma(\bar{S}_{|\overline{M_x}})$ is countable and closed in $\mathbb{C}$, $\sigma(\bar{S}_{|\overline{M_x}})$ is not a perfect set (see [7], theorem 2.43), and hence $\sigma(\bar{S}_{|\overline{M_x}})$ has an isolated point λ_0 which is an eigenvalue. Hence, there exists a non zero $z \in \overline{M_x}$ such that $\overline{S(n)}z = \lambda_0^n z$, $\forall n \in \mathbb{N}$.

From $\lim_{\alpha \downarrow 0} \alpha \hat{x}(\lambda_0 e^\alpha) = M_{\lambda_0}(x)$, we have

$$\lim_{\alpha \downarrow 0} \alpha \sum_{k=0}^{+\infty} (\lambda_0 e^\alpha)^{-k-1} \overline{S(k)x} = \overline{M_{\lambda_0}(x)} = \bar{0}.$$

Moreover, $\lim_{\alpha \downarrow 0} \alpha \sum_{k=0}^{+\infty} (\lambda_0 e^\alpha)^{-k-1} \overline{S(k)}g = \bar{0}, \quad \forall g \in \overline{M_x}$.

Observe that the following formula holds:

$$\alpha \sum_{k=0}^{+\infty} (\lambda_0 e^\alpha)^{-k-1} \overline{S(k)}z = \frac{\alpha}{\lambda_0(e^\alpha - 1)} z.$$

Letting $\alpha \downarrow 0$, we have $z = \bar{0}$. This contradiction proves the result. □

3. Asymptotically almost periodic solutions of discrete system on the half-line

Let A be a linear bounded operator on $\mathbb{X}$ and $f \in l_\infty^+(\mathbb{X})$, and consider the abstract inhomogeneous Cauchy problem

$$x(n+1) = Ax(n) + f(n), \forall n \in \mathbb{N}. \tag{3}$$

THEOREM 3.1 *Suppose that* $\sigma_\Gamma(A) := \sigma(A) \cap \Gamma$ *is countable and* $f \in AAP(\mathbb{N}, \mathbb{X})$, x *is a bounded solution of equation* (3) *which is totally ergodic. Then,* $x \in AAP(\mathbb{N}, \mathbb{X})$.

Proof. Fomula (3) is equivalent to

$$Sx = A_1 x + f,$$

where $A_1 x := \{Ax(n)\}_{n\in\mathbb{N}}$. Hence $\bar{S}\bar{x} = A_1\bar{x}$. Therefore, for any $\lambda \in \mathbb{C}$ with $|\lambda| > 1$ we get

$$(\lambda - A_1)\widehat{x_{AP}}(\lambda) = (\lambda - A_1)\sum_{n=0}^{\infty} \lambda^{-n-1}\overline{S(n)}\bar{x}$$

$$= \sum_{n=0}^{\infty} \lambda^{-n}\overline{S(n)}\bar{x} - \sum_{n=0}^{\infty} \lambda^{-n-1}\overline{S(n)}\bar{x} \quad = \bar{x},$$

which shows that $\widehat{x_{AP}}$ has a holomorphic extension $R(\lambda, A_1)$ at any $\lambda \notin \sigma_\Gamma(A_1) = \sigma_\Gamma(A)$. Thus we have $\sigma_{AP}^+(x) \subset \sigma_\Gamma(A)$. It follows from Theorem 2.14, $x \in AAP(\mathbb{N}, \mathbb{X})$.

□

Example 3.2 Let $X := \mathbb{R}^m$, $A \in M_n(\mathbb{R})$, $f = 0$ and $\sigma(A)$ consists m eigenvalues t_k, $k \in \{1,2,\ldots m\}$ which satisfies the following conditions:

(i) $t_k \neq t_s$ $\forall k \neq s$ and k, $s \in \{1,2,\ldots,m\}$,
(ii) $|t_k| \leq 1, \forall k \in \{1, 2, \ldots, m\}$.

Then, the equation

$$x(n+1) = Ax(n), \forall n \in \mathbb{N}$$

satisfies all the conditions of Theorem 3.1.

Proof. It is easy to show that

$$x(n) = \sum_{i=0}^{m} t_i^n P_i x(0), \quad \forall n \in \mathbb{N},$$

where $P_i = \prod_{j=1, j\neq i}^{m} \frac{A - t_j}{t_i - t_j}$, $i \in \{1, 2, \ldots, m\}$.

To prove the example, we solve the two following cases.

- $x \in c_0$ is totally ergodic.

  Let a given $\varepsilon > 0$, $\lambda_0 \in \Gamma$. Choose $n_0 : \|S(n)x\| \le \varepsilon$, $\forall n \ge n_0$, we see that

$$\left\|\alpha\sum_{n=0}^{\infty}(\lambda_0 e^{\alpha})^{-n-1}S(n)x\right\| \le \left\|\alpha\sum_{n=0}^{n_0}(\lambda_0 e^{\alpha})^{-n-1}S(n)x\right\| + \left\|\alpha\sum_{n=n_0+1}^{\infty}(\lambda_0 e^{\alpha})^{-n-1}S(n)x\right\|$$
$$\le |\alpha|\sum_{n=0}^{n_0}(e^{\alpha})^{-n-1}\|x\| + \frac{\varepsilon|\alpha|e^{-\alpha(n_0+1)}}{e^{\alpha}-1}.$$

  Letting $\alpha \downarrow 0$, it is obvious to show that x is totally ergodic.

- $x = \{t_0^n c\}_{n\in\mathbb{N}} : t_0 \in \Gamma$, $c \in \mathbb{X}$ is totally ergodic.

  We have

$$\alpha\sum_{n=0}^{\infty}(\lambda_0 e^{\alpha})^{-n-1}S(n)x = \alpha\sum_{n=0}^{\infty}(\lambda_0 e^{\alpha})^{-n-1}t_0^n x \quad = \frac{\alpha t_0}{\lambda_0 e^{\alpha}(t_0 - \lambda_0 e^{\alpha})}x.$$

  Hence, x is totally ergodic.

Finally, the sum of two totally ergodic sequences is totally ergodic, so the example is proved.

□

4. Applications to evolution equation

Although the result of previous section should have independent interest, we now discuss several applications of our result to study the asymptotically almost periodic solutions of evolution equations.

4.1 Asymptotically almost periodic solutions to periodic evolution equations

We consider in this section the following equation

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = A(t)x(t) + f(t), \quad t \in \mathbb{R}^+, \tag{5}$$

where $A(t)$ is a (in general unbounded) linear operator on $\mathbb{X}$ which is periodic and f is asymptotically almost periodic. For more details, we refer to [1].

We now consider in the subsection condition for the existence of mild asymptotically almost periodic solution to equation (5). Once equation (5) is well-posed, this problem is actually reduced to find conditions for the existence of asymptotically almost periodic solutions to the following more general equation

$$x(t) = U(t,s)x(s) + \int_s^t U(t,\xi)f(\xi)\,\mathrm{d}\xi,\ \forall t \ge s \ge 0, \tag{6}$$

where $(U(t, s))_{t\ge s\ \ge\ 0}$ is a 1-evolution process on the half-line, i.e. it satisfies conditions as stated in the following definition.

DEFINITION 4.1 *A family of bounded linear operators* $(U(t,s))_{t\geq s\geq 0}$ *from a Banach space* $\mathbb{X}$ *to itself is called* 1*-periodic strongly continuous evolutionary process on the half-line if the following conditions are satisfied:*

(i) $U(t,t) = I, \quad \forall t \in \mathbb{R}^+,$
(ii) $U(t,s)U(s,r) = U(t,r), \quad \forall t \geq s \geq r \geq 0,$
(iii) *The map* $(t,s) \to U(t,s)x$ *is continuous for every fixed* $x \in \mathbb{X}$,
(iv) $U(t+1, s+1) = U(t,s), \quad \forall t \geq s \geq 0,$
(v) $\|U(t,s)\| < Ne^{\omega(t-s)}, \quad \forall t \geq s \geq 0,$ *for some positive* N, ω *independent of* t, s.

LEMMA 4.2 *Let* $(U(t,s))_{t\geq s\geq 0}$ *be a* 1*-periodic strongly continuous evolutionary process and* f *be asymptotically almost periodic. Suppose that* u *is a solution on the half-line of equation* (5)*. Then, if the sequence* $\{u(n)\}_{n\in\mathbb{N}}$ *is asymptotically almost periodic, the solution* u *is asymptotically almost periodic as well.*

Proof. We define a function w from $\mathbb{R}^+$ to $\mathbb{X}$ as following

$$w(t) := su(n) + (1-s)u(n+1), \quad t = sn + (1-s)(n+1), \quad s \in [0,1],\ n \in \mathbb{N}.$$

It is obvious that the function is asymptotically almost periodic. Hence, the function $g(t) := (w(t),\ f(t))$ defined on $\mathbb{R}^+$ is asymptotically almost periodic (see [1], page 305, Theorem 4.7.4) and the sequence $\{g(n)\}_{n\in\mathbb{N}}$ is asymptotically almost periodic, too.

For each given $\varepsilon > 0$, there is a positive real number μ such that the following set

$$Q := Q_{\varepsilon,\mu}(g) \cap \mathbb{N}$$

is relatively dense in $\mathbb{Z}$, where

$$Q_{\varepsilon,\mu}(g) := \left\{ \tau \in \mathbb{R}^+ : \sup_{t\in\mathbb{R}^+, t\geq\mu} \|g(t+\tau) - g(t)\| \leq \varepsilon \right\}.$$

Hence, for every $m \in Q$

$$\|f(m+t) - f(t)\| \leq \varepsilon, \quad \forall t \geq \mu, \quad \|u(m+n) - u(n)\| \leq \varepsilon, \quad \forall n \in \mathbb{N},\ n \geq \mu.$$

Since u is a solution of equation (5), we have

$$\begin{aligned} \|u(n+m+s) - u(n+s)\| \leq\ & \|U(s,0)(u(n+m) - u(n))\| \\ & + \left\| \int_0^s U(s,\xi)(f(n+m+\xi) - f(n+\xi))\, \mathrm{d}\xi \right\| \\ \leq\ & \mathbb{N}e^{\omega}\|u(n+m) - u(n)\| + N\frac{e^{\omega}}{\omega} \sup_{t\in\mathbb{R}^+} \|f(m+t) - f(t)\| \end{aligned}$$

for all $n \geq \mu$, $s \in [0,1]$.

Clearly $m \in Q_{\varepsilon_1,\mu}(u)$ for $\varepsilon_1 := \varepsilon\big(1 + (1/\omega)\big)Ne^{\omega}$. Moreover, it follows from Theorem 4.7.5 (see [1]) that u is asymptotically almost periodic.

□

Consider the function

$$g(t) := \int_t^{t+1} U(t+1,\xi) f(\xi)\,\mathrm{d}\xi, \quad t \in \mathbb{R}^+.$$

We can easily verify that the sequence $\{g(n)\}_{n\in\mathbb{N}}$ is asymptotically almost periodic.

THEOREM 4.3 *Suppose that equation* (5) *has a bounded solution* $x(t)$, $\sigma_\Gamma(U(1,0))$ *is countable, the sequence* $\{x(n)\}_{n\in\mathbb{N}}$ *is totally ergodic. Then,* $x(t)$ *is asymptotically almost periodic.*

Proof. From the 1-periodicity of the process $(U(t,s))_{t\geq s\geq 0}$, we can deal with the discrete equation

$$x(n+1) = U(1,0)x(n) + g(n).$$

By the sequence $\{g(n)\}_{n\in\mathbb{N}}$ is asymptotically almost periodic, it follows from Theorem 3.1 and Proposition 4.2 that the bounded solution $x(t)$ is asymptotically almost periodic.

□

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References

[1] Arendt, W., Batty, K., Hieber, M. and Neubrander, F., 2001, *Vector-Valued Laplace Transforms and Cauchy Problems* (Monogrphs in mathematics) (Basel-Boston-Berlin: Birkhäuser Verlag), Vol. 96.
[2] Fink, M., 1974, *Almost Periodic Differential Equation. Lecture Notes in Math* (Berlin: Springer-Verlag).
[3] Batty, K., Hutter, W. and Räbiger, F., 1999, Almost periodicity of mild solution of inhomogeneous periodic Cauchy problems. *Journal of Differential Equations*, **156**, 309–327.
[4] Levitan, B.M. and Zhikov, V.V., 1978, *Almost Periodic Functions and Differential Equations* (Moscow University Publishing House), (English translation by Cambridge University Press (1982)).
[5] Murakami, S., Naito, T. and Van Minh, Nguyen, 2000, Evolution semigroups and sums of commuting operators: a new approach to the admissibility theory of function spaces. *Journal of Differential Equations*, **164**, 240–285.
[6] Naito, T., Van Minh, N., Miyazaki, R. and Hamaya, Y., 2001, Boundedness and almost periodicity in dynamical systems. *Journal of Difference Equations and Applications*, **7**, 507–527.
[7] Rudin, W., 1954, *Principles of Mathematical Analysis* (New York: McGraw-Hill).