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Ramification of the Gauss map and the total curvature of a complete minimal surface

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In this article, we study the relations between the ramifications of the Gauss map and the total curvature of a complete minimal surface. More precisely, we introduce some conditions on the ramifications of the Gauss map of a complete minimal surface M to show that M has finite total curvature.

1. Introduction

In 1988, Fujimoto [3] proved Nirenberg's conjecture that if M is a complete non-flat minimal surface in $\mathbb{R}^3$, then its Gauss map can omit at most 4 points, and there are a number of examples showing that the bound is sharp (see [12, pp. 72–74]). He [4] also extended that result to the Gauss map of complete minimal surfaces in $\mathbb{R}^m$. After that, in 1990, Mo–Osserman [10] showed an interesting improvement of Fujimoto's result by proving that a complete minimal surface in $\mathbb{R}^3$ whose Gauss map assumes five values only a finite number of times has finite total curvature. We note that a complete minimal surface with finite total

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curvature to be called an algebraic minimal surface. After that, Mo [9] extended that result to the complete minimal surface in $\mathbb{R}^m$ $(m > 3)$.

On the other hand, in 1993, M. Ru [13] refined the results of Fujimoto by studying the Gauss map of minimal surfaces in $\mathbb{R}^m$ with ramification. Many results related to this problem were studied (see Jin–Ru [7], Kawakami–Kobayashi–Miyaoka [8], Ha [5], Dethloff–Ha [1] and Dethloff–Ha–Thoan [2] for examples).

A natural question is whether we may show a relation between of the ramification of the Gauss map and the total curvature of a complete minimal surface. The main purpose of this article is to give an affirmative answer for this question. For the purpose of this article, we recall some definitions.

Let $x = (x_0, \cdots, x_{m-1}) : M \to \mathbb{R}^m$ be a (smooth, oriented) minimal surface immersed in $\mathbb{R}^m$. Then M has the structure of a Riemann surface and any local isothermal coordinate (ξ_1, ξ_2) of M gives a local holomorphic coordinate $z = \xi_1 + \sqrt{-1}\xi_2$. The (generalized) Gauss map of x is defined to be

$$g : M \to Q_{m-2}(\mathbb{C}) \subset \mathbb{P}^{m-1}(\mathbb{C}), g(z) = (\frac{\partial x_0}{\partial z} : \cdots : \frac{\partial x_{m-1}}{\partial z}),$$

where

$$Q_{m-2}(\mathbb{C}) = \{(w_0 : \cdots : w_{m-1}) | w_0^2 + \cdots + w_{m-1}^2 = 0\} \subset \mathbb{P}^{m-1}(\mathbb{C}).$$

By the assumption of minimality of M, g is a holomorphic map of M into $Q_{m-2}(\mathbb{C})$.

One says that g is ramified over a hyperplane $H = \{(w_0 : \cdots : w_{m-1}) \in \mathbb{P}^{m-1}(\mathbb{C}) : a_0 w_0 + \cdots + a_{m-1} w_{m-1} = 0\}$ *with multiplicity at least* e if all the zeros of the function $(g, H) := a_0 g_0 + \cdots + a_{m-1} g_{m-1}$ have orders at least e, where $g = (g_0 : \cdots : g_{m-1})$. If the image of g omits H, one will say that g is *ramified over* H *with multiplicity* ∞.

The main purpose of this article is to prove the following:

Theorem 1. *Let M be a complete minimal surface in $\mathbb{R}^m$ and K be a compact subset in M. Assume that the generalized Gauss map g of M is k-non-degenerate (that is $g(M)$ is contained in a k-dimensional linear subspace in $\mathbb{P}^{m-1}(\mathbb{C})$, but none of lower dimension), $1 \leq k \leq m-1$. If there are q hyperplanes $\{H_j\}_{j=1}^q$ in N-subgeneral position in $\mathbb{P}^{m-1}(\mathbb{C})$, $(N \geq m-1)$ such that g is ramified over H_j with multiplicity at least m_j on $M \setminus K$ for each j and*

$$\sum_{j=1}^{q} (1 - \frac{k}{m_j}) > (k+1)(N - \frac{k}{2}) + (N+1), \tag{1.1}$$

then M has finite total curvature.

In particular, if $\{H_j\}_{j=1}^q$ are in general position in $\mathbb{P}^{m-1}(\mathbb{C})$ and

$$\sum_{j=1}^{q} (1 - \frac{m-1}{m_j}) > \frac{m(m+1)}{2}, \tag{1.2}$$

then M must have finite total curvature.

When $m = 3$, we can identify $\mathbb{Q}_1(\mathbb{C})$ with $\mathbb{P}^1(\mathbb{C})$. So we can get a better result as the following:

Theorem 2. *Let M be a complete minimal surface in $\mathbb{R}^3$ and q distinct points $a^j, \ldots, a^q$ in $\mathbb{P}^1(\mathbb{C})$. Suppose that the Gauss map g of M is ramified over a^j with multiplicity at least m_j for each $j = 1, \cdots, q$ outside a compact subset K of M. Then M has finite total curvature if*

$$\sum_{j=1}^{q}\left(1-\frac{1}{m_j}\right)>4. \tag{1.3}$$

We now give some applications of Theorem 1 and Theorem 2 by using them to prove some previous results of Mo–Osserman [10], Mo [9] and Ru [13]:

Theorem 3. *([10, Theorem 1]) Let M be a complete minimal surface in $\mathbb{R}^3$. If Gauss map g takes on five distinct points in $\mathbb{P}^1(\mathbb{C})$ only a finite number of times. Then M has finite total curvature.*

Proof. Assume that the Gauss map g takes on five distinct points $a^1,\dots,a^5$ in $\mathbb{P}^1(\mathbb{C})$ only a finite number of times, we can choose a compact subset K of M which contains $g^{-1}(a^1),\dots,g^{-1}(a^5)$. So the Gauss map g will omit $a^1,\dots,a^5$ outside K (i.e. g ramifies over $a^1,\dots,a^5$ with multiplicity ∞). We now apply the Theorem 2 to show that M has finite total curvature. Theorem 3 is proved. □

Theorem 4. *([9]) Let M be a complete non-degenerate minimal surface in $\mathbb{R}^m$ such that its generalized Gauss map g intersects only a finite number of times the hyperplanes $\{H_j\}_{j=1}^q$ in $\mathbb{P}^{m-1}(\mathbb{C})$ in general position. If $q>m(m+1)/2$ then M must have finite total curvature.*

Proof. Indeed, if we assume that the Gauss map g intersects q hyperplanes $H_1,\dots,H_q$ in $\mathbb{P}^{m-1}(\mathbb{C})$ in general position only a finite number of times, we can choose a compact subset K of M which contains $g^{-1}(H_1),\dots,g^{-1}(H_q)$. So the Gauss map g will omit $H_1,\dots,H_q$ outside K (i.e. g ramifies over $H_1,\dots,H_q$ with multiplicity ∞). We now apply the Theorem 1 to show that M has finite total curvature. Theorem 4 is proved. □

Theorem 5. *([13, Theorem 2]) Let M be a non-flat complete minimal surface in $\mathbb{R}^3$. If there are q $(q>4)$ distinct points $a^1,\dots,a^q\in\mathbb{P}^1(\mathbb{C})$ such that the Gauss map g of M is ramified over a^j with multiplicity at least m_j for each j, then $\sum_{j=1}^q(1-\frac{1}{m_j})\le 4$.*

Proof. We set K to be an empty set in a non-flat complete minimal surface M. So if (1.3) is correct, by using Theorem 2, we show that the minimal surface M has finite total curvature. Now, by the completeness of M we have M to be an algebraic minimal surface. Thanks to Theorem 3.3 in [8], we obtain $\sum_{j=1}^q(1-\frac{1}{m_j})<4$. This gives a contradiction. Thus, Theorem 5 is proved. □

Theorem 6. *([13, Theorem 1]) For any complete minimal surface M immersed in $\mathbb{R}^m$ with its Gauss map g. Assume that the generalized Gauss map g of M is k-non-degenerate, $1\le k\le m-1$. If there are q hyperplanes $\{H_j\}_{j=1}^q$ in general position in $\mathbb{P}^{m-1}(\mathbb{C})$ such that g is ramified over H_j with multiplicity at least m_j on M for each j. Then*

$$\sum_{j=1}^{q}\left(1-\frac{k}{m_j}\right)\le (k+1)\left(m-\frac{k}{2}-1\right)+m. \tag{1.4}$$

In particular, Let $\{H_j\}_{j=1}^q$ be q hyperplanes in general position in $\mathbb{P}^{m-1}(\mathbb{C})$. If g is ramified over H_j with multiplicity at least m_j for each j and

$$\sum_{j=1}^{q}\left(1-\frac{m-1}{m_j}\right)>\frac{m(m+1)}{2}$$

then M is flat, or equivalently, g is constant.

Proof. Assume M is a non-flat complete minimal surface and K is an empty set. So if (1.4) is not correct, by using Theorem 1 for the case $N = m-1$, we show that the minimal surface M has finite total curvature. Now, by the completeness of M we have M to be an algebraic minimal surface. Thanks to the proof of Theorem 3.1 in [7], we can obtain

$$\sum_{j=1}^{q}(1-\frac{k}{m_j}) < (k+1)(m-\frac{k}{2}-1)+m.$$

This gives a contradiction. So M must be flat. Theorem 6 is proved. □

2. Auxiliary lemmas

Let f be a linearly non-degenerate holomorphic map of $\Delta_R := \{z \in \mathbb{C} : |z| < R\}$ into $\mathbb{P}^k(\mathbb{C})$, where $0 < R \le +\infty$. Take a reduced representation $f = (f_0 : \cdots : f_k)$. Then $F := (f_0, \cdots, f_k) : \Delta_R \to \mathbb{C}^{k+1} \setminus \{0\}$ is a holomorphic map with $\mathbb{P}(F) = f$. Consider the holomorphic map

$$F_p = (F_p)_z := F^{(0)} \wedge F^{(1)} \wedge \cdots \wedge F^{(p)} : \Delta_R \longrightarrow \wedge^{p+1}\mathbb{C}^{k+1}$$

for $0 \le p \le k$, where $F^{(0)} := F = (f_0, \cdots, f_k)$ and $F^{(l)} = (F^{(l)})_z := (f_0^{(l)}, \cdots, f_k^{(l)})$ for each $l = 0, \cdots, k$, and where the l-th derivatives $f_i^{(l)} = (f_i^{(l)})_z$, $i = 0, \cdots, k$, are taken with respect to z. (Here and for the rest of this paper the index $|_z$ means that the corresponding term is defined by using differentiation with respect to the variable z, and in order to keep notations simple, we usually drop this index if no confusion is possible.) The norm of F_p is given by

$$|F_p| := \left(\sum_{0 \le i_0 < \cdots < i_p \le k} \left| W(f_{i_0}, \cdots, f_{i_p}) \right|^2 \right)^{\frac{1}{2}},$$

where $W(f_{i_0}, \cdots, f_{i_p}) = W_z(f_{i_0}, \cdots, f_{i_p})$ denotes the Wronskian of $f_{i_0}, \cdots, f_{i_p}$ with respect to z.

Proposition 7. *([4, Proposition 2.1.6]) For two holomorphic local coordinates z and ξ and a holomorphic function $h : \Delta_R \to \mathbb{C}$, the following holds:*

a) $W_\xi(f_0, \cdots, f_p) = W_z(f_0, \cdots, f_p) \cdot (\frac{dz}{d\xi})^{p(p+1)/2}$.
b) $W_z(hf_0, \cdots, hf_p) = W_z(f_0, \cdots, f_p) \cdot (h)^{p+1}$.

Proposition 8. *([4, Proposition 2.1.7]) For holomorphic functions $f_0, \ldots, f_p : \Delta_R \to \mathbb{C}$ the following conditions are equivalent:*

(i) *$f_0, \ldots, f_p$ are linearly dependent over $\mathbb{C}$.*
(ii) *$W_z(f_0, \cdots, f_p) \equiv 0$ for some (or all) holomorphic local coordinate z.*

We now take a hyperplane H in $\mathbb{P}^k(\mathbb{C})$ given by

$$H : \overline{c}_0\omega_0 + \cdots + \overline{c}_k\omega_k = 0\,,$$

with $\sum_{i=0}^{k} |c_i|^2 = 1$. We set

$$F_0(H) := F(H) := \overline{c}_0 f_0 + \cdots + \overline{c}_k f_k$$

and

$$|F_p(H)| = |(F_p)_z(H)| := \left(\sum_{0\le i_1<\cdots<i_p\le k} \left| \sum_{l\neq i_1,\dots,i_p} \overline{c}_l W(f_l, f_{i_1}, \cdots, f_{i_p}) \right|^2 \right)^{\frac{1}{2}},$$

for $1 \le p \le k$. We note that by using Proposition 7, $|(F_p)_z(H)|$ is multiplied by a factor $|\frac{dz}{d\xi}|^{p(p+1)/2}$ if we choose another holomorphic local coordinate ξ, and it is multiplied by $|h|^{p+1}$ if we choose another reduced representation $f = (hf_0 : \cdots : hf_k)$ with a nowhere zero holomorphic function h. Finally, for $0 \le p \le k$, set the p-th contact function of f for H to be $\phi_p(H) := \dfrac{|F_p(H)|^2}{|F_p|^2} = \dfrac{|(F_p)_z(H)|^2}{|(F_p)_z|^2}$.

We next consider q hyperplanes $H_1, \dots, H_q$ in $\mathbb{P}^k(\mathbb{C})$ given by

$$H_j : \langle \omega, A_j \rangle \equiv \overline{c}_{j0}\omega_0 + \cdots + \overline{c}_{jk}\omega_k \quad (1 \le j \le q)$$

where $A_j := (c_{j0}, \cdots, c_{jk})$ with $\sum_{i=0}^{k} |c_{ji}|^2 = 1$.

Assume now $N \ge k$ and $q \ge N+1$. For $R \subseteq Q := \{1, 2, \cdots, q\}$, denote by $d(R)$ the dimension of the vector subspace of $\mathbb{C}^{k+1}$ generated by $\{A_j; j \in R\}$.

The hyperplanes $H_1, \dots, H_q$ are said to be in N-subgeneral position if $d(R) = k+1$ for all $R \subseteq Q$ with $\sharp(R) \ge N+1$, where $\sharp(A)$ means the number of elements of a set A. In the particular case $N = k$, these are said to be in general position.

Theorem 9. *([4, Theorem 2.4.11]) For given hyperplanes $H_1, \dots, H_q$ $(q > 2N - k + 1)$ in $\mathbb{P}^k(\mathbb{C})$ located in N-subgeneral position, there are some rational numbers $\omega(1), \dots, \omega(q)$ and θ satisfying the following conditions:*

(i) $0 < \omega(j) \le \theta \le 1$ $(1 \le j \le q)$,
(ii) $\sum_{j=1}^{q} \omega(j) = k + 1 + \theta(q - 2N + k - 1)$,
(iii) $\frac{k+1}{2N-k+1} \le \theta \le \frac{k+1}{N+1}$,
(iv) *If $R \subset Q$ and $0 < \sharp(R) \le n+1$, then $\sum_{j\in R} \omega(j) \le d(R)$.*

Constants $\omega(j)$ $(1 \le j \le q)$ and θ with the properties of Theorem 9 are called Nochka weights and a Nochka constant for $H_1, \dots, H_q$ respectively. Related to Nochka weights, we have the following.

Proposition 10. *([4, Lemma 3.2.13]) Let f be a non-degenerate holomorphic map of a domain in $\mathbb{C}$ into $\mathbb{P}^k(\mathbb{C})$ with reduced representation $f = (f_0 : \cdots : f_k)$ and let $H_1, \dots, H_q$ be hyperplanes located in N-subgeneral position $(q > 2N - k + 1)$ with Nochka weights $\omega(1), \dots, \omega(q)$ respectively. Then,*

$$\nu_\phi + \sum_{j=1}^{q} \omega(j) \cdot \min(\nu_{(f,H_j)}, k) \ge 0,$$

where $\phi = \dfrac{|F_k|}{\Pi_{j=1}^{q} \mid F(H_j) \mid^{\omega(j)}}$ and ν_ϕ is the divisor of ϕ.

Lemma 11. *([2, Lemma 9]) Let $f = (f_0 : \cdots : f_k) : \Delta_R \to \mathbb{P}^k(\mathbb{C})$ be a non-degenerate holomorphic map, $H_1, \dots, H_q$ be hyperplanes in $\mathbb{P}^k(\mathbb{C})$ in N-subgeneral position ($N \ge k$ and $q > 2N - k + 1$), and $\omega(j)$ be their Nochka weights. If*

$$\gamma := \sum_{j=1}^{q} \omega(j)(1 - \frac{k}{m_j}) - (k+1) > 0$$

and f is ramified over H_j with multiplicity at least $m_j \geq k$ for each j, $(1 \leq j \leq q)$, then for any positive ϵ with $\gamma > \epsilon\sigma_{k+1}$ there exists a positive constant C, depending only on $\epsilon, H_j, m_j, \omega(j)$ $(1 \leq j \leq q)$, such that

$$|F|^{\gamma-\epsilon\sigma_{k+1}} \frac{|F_k|^{1+\epsilon} \prod_{j=1}^{q} \prod_{p=0}^{k-1} |F_p(H_j)|^{\epsilon/q}}{\prod_{j=1}^{q} |F(H_j)|^{\omega(j)(1-\frac{k}{m_j})}} \leqslant C(\frac{2R}{R^2-|z|^2})^{\sigma_k+\epsilon\tau_k},$$

where $\sigma_p = p(p+1)/2$ for $0 \leq p \leq k$ and $\tau_k = \sum_{p=0}^{k} \sigma_p$.

In particular, we have the following version for the case one dimension.

Lemma 12. *([1, Lemma 8]). For every δ with $q - 2 - \sum_{j=1}^{q} \frac{1}{m_j} > q\delta > 0$ and f which is ramified over $a^j \in \mathbb{P}^1(\mathbb{C})$ with multiplicity at least m_j for each j $(1 \leq j \leq q)$, there exists a positive constant C such that*

$$\frac{||f||^{q-2-\sum_{j=1}^{q}\frac{1}{m_j}-q\delta}|W(f_0,f_1)|}{\Pi_{j=1}^{q}|F_j|^{1-\frac{1}{m_j}-\delta}} \leq C\frac{2R}{R^2-|z|^2}.$$

Lemma 13. *([4, Theorem 3.3.15]). Let $f : \Delta_{s,\infty}(=\mathbb{C}-\Delta_s) \to \mathbb{P}^n(\mathbb{C})$ be a nonconstant holomorphic map and let $H_1, \ldots, H_q$ be distinct q hyperplanes in N-subgeneral position. Assume that f has an essential singularity at ∞ in the particular case $s > 0$, and is ramified over H_j $(j = 1, \cdots, q)$ with multiplicity at least m_j for each j. Then*

$$\sum_{j=1}^{q}(1-\frac{n}{m_j}) \leq 2N-n+1.$$

We finally will need the following result on completeness of open Riemann surfaces with conformally flat metrics due to Fujimoto:

Lemma 14. *([4, Lemma 1.6.7]). Let $d\sigma^2$ be a conformal flat metric on an open Riemann surface M. Then for every point $p \in M$, there is a holomorphic and locally biholomorphic map Φ of a disk (possibly with radius ∞) $\Delta_{R_0} := \{w : |w| < R_0\}$ $(0 < R_0 \leq \infty)$ onto an open neighborhood of p with $\Phi(0) = p$ such that Φ is a local isometry, namely the pull-back $\Phi^*(d\sigma^2)$ is equal to the standard (flat) metric on Δ_{R_0}, and for some point a_0 with $|a_0| = 1$, the Φ-image of the curve*

$$L_{a_0} : w := a_0 \cdot s \ (0 \leq s < R_0)$$

is divergent in M (i.e. for any compact set $K \subset M$, there exists an $s_0 < R_0$ such that the Φ-image of the curve $L_{a_0} : w := a_0 \cdot s$ $(s_0 \leq s < R_0)$ does not intersect K).

3. The proof of Theorem 1

Proof. For the convenience of the reader, we first recall some notations on the Gauss map of minimal surfaces in $\mathbb{R}^m$. Let M be a complete immersed minimal surface in $\mathbb{R}^m$. Take an immersion $x = (x_0, \cdots, x_{m-1}) : M \to \mathbb{R}^m$. Then M has the structure of a Riemann surface and any local isothermal coordinate (ξ_1, ξ_2) of M gives a local holomorphic coordinate $z = \xi_1 + \sqrt{-1}\xi_2$. The generalized Gauss map of x is defined to be

$$g : M \to \mathbb{P}^{m-1}(\mathbb{C}), g = \mathbb{P}(\frac{\partial x}{\partial z}) = (\frac{\partial x_0}{\partial z} : \cdots : \frac{\partial x_{m-1}}{\partial z}).$$

Since $x : M \to \mathbb{R}^m$ is immersed,

$$G = G_z := (g_0, \cdots, g_{m-1}) = ((g_0)_z, \cdots, (g_{m-1})_z) = (\frac{\partial x_0}{\partial z}, \cdots, \frac{\partial x_{m-1}}{\partial z})$$

is a (local) reduced representation of g, and since for another local holomorphic coordinate ξ on M we have $G_\xi = G_z \cdot (\frac{dz}{d\xi})$, g is well defined (independently of the (local) holomorphic coordinate). Moreover, if ds^2 is the metric on M induced by the standard metric on $\mathbb{R}^m$, we have

$$ds^2 = 2|G_z|^2|dz|^2. \tag{3.1}$$

Finally since M is minimal, g is a holomorphic map.

Since by hypothesis of the Theorem 1, g is k-non-degenerate $(1 \le k \le m-1)$ without loss of generality, we may assume that $g(M) \subset \mathbb{P}^k(\mathbb{C})$; then

$$g : M \to \mathbb{P}^k(\mathbb{C}), g = \mathbb{P}(\frac{\partial x}{\partial z}) = (\frac{\partial x_0}{\partial z} : \cdots : \frac{\partial x_k}{\partial z})$$

is linearly non-degenerate in $\mathbb{P}^k(\mathbb{C})$ (so in particular g is not constant) and the other facts mentioned above still hold.

Now the proof of Theorem 1 will be given in six steps:

Step 1: Let H_j $(j = 1, \cdots, q)$ be $q(\ge N+1)$ hyperplanes in $\mathbb{P}^{m-1}(\mathbb{C})$ in N-subgeneral position $(N \ge m-1 \ge k)$. Then $H_j \cap \mathbb{P}^k(\mathbb{C})$ $(j = 1, \cdots, q)$ are q hyperplanes in $\mathbb{P}^k(\mathbb{C})$ in N-subgeneral position. Let each $H_j \cap \mathbb{P}^k(\mathbb{C})$ be represented as

$$H_j \cap \mathbb{P}^k(\mathbb{C}) : \overline{c}_{j0}\omega_0 + \cdots + \overline{c}_{jk}\omega_k = 0$$

with $\sum_{i=0}^k |c_{ji}|^2 = 1$.

Set

$$G(H_j) = G_z(H_j) := \overline{c}_{j0}g_0 + \cdots + \overline{c}_{jk}g_k.$$

We will now, for each contact function $\phi_p(H_j)$ of g for each a hyperplane H_j, choose one of the components of the numerator $|((G_z)_p)_z(H_j)|$ which is not identically zero: More precisely, for each j, p $(1 \le j \le q, 1 \le p \le k)$, we can choose $i_1, \ldots, i_p$ with $0 \le i_1 < \cdots < i_p \le k$ such that

$$\psi(G)_{jp} = (\psi(G_z)_{jp})_z := \sum_{l \ne i_1, \ldots, i_p} \overline{c}_{jl} W_z(g_l, g_{i_1}, \cdots, g_{i_p}) \not\equiv 0,$$

(indeed, otherwise, we have $\sum_{l \ne i_1, \ldots, i_p} \overline{c}_{jl} W(g_l, g_{i_1}, \cdots, g_{i_p}) \equiv 0$ for all $i_1, \ldots, i_p$, so $W(\sum_{l \ne i_1, \ldots, i_p} \overline{c}_{jl} g_l, g_{i_1}, \cdots, g_{i_p}) \equiv 0$ for all $i_1, \ldots, i_p$, which contradicts the non-degeneracy of g in $\mathbb{P}^k(\mathbb{C})$. Alternatively we simply can observe that in our situation none of the contact functions vanishes identically). We still set $\psi(G)_{j0} = \psi(G_z)_{j0} := G(H_j)(\not\equiv 0)$, and we also note that $\psi(G)_{jk} = ((G_z)_k)_z$. Since the $\psi(G)_{jp}$ are holomorphic, so they have only isolated zeros.

Finally we put for later use the transformation formulas for all the terms defined above, which are obtained by using Proposition 7: For local holomorphic coordinates z and ξ on M we have:

$$G_\xi = G_z \cdot (\frac{dz}{d\xi})\,, \tag{3.2}$$

$$G_\xi(H) = G_z(H) \cdot (\frac{dz}{d\xi})\,, \tag{3.3}$$

$$((G_\xi)_k)_\xi = ((G_z)_k)_z \cdot (\frac{dz}{d\xi})^{k+1+\frac{k(k+1)}{2}} = ((G_z)_k)_z(\frac{dz}{d\xi})^{\sigma_{k+1}}\,, \tag{3.4}$$

$$(\psi(G_\xi)_{jp})_\xi = (\psi(G_z)_{jp})_z \cdot (\frac{dz}{d\xi})^{p+1+\frac{p(p+1)}{2}} = (\psi(G_z)_{jp})_z \cdot (\frac{dz}{d\xi})^{\sigma_{p+1}}\,,\ (0 \le p \le k)\,. \tag{3.5}$$

Moreover, we also will need the following transformation formulas for mixed variables:

$$((G_\xi)_k)_\xi = ((G_\xi)_k)_z \cdot (\frac{dz}{d\xi})^{\frac{k(k+1)}{2}} = ((G_\xi)_k)_z(\frac{dz}{d\xi})^{\sigma_k}\,, \tag{3.6}$$

$$(\psi(G_\xi)_{jp})_\xi = (\psi(G_\xi)_{jp})_z \cdot (\frac{dz}{d\xi})^{\frac{p(p+1)}{2}} = (\psi(G_\xi)_{jp})_z \cdot (\frac{dz}{d\xi})^{\sigma_p}\,,\ (0 \le p \le k)\,. \tag{3.7}$$

We next observe that we may also assume

$$m_j > k\,,\ j = 1, \cdots, q\,. \tag{3.8}$$

In fact, if this does not hold for all $j = 1, \ldots, q$, we just drop the H_j for which it does not hold, and remain with $\tilde{q} < q$ such hyperplanes. By hypothesis (1.1), $\widetilde{q} \ge N + 1$ and the $\widetilde{q}$ hyperplanes thus obtained are still in N-subgeneral position in $\mathbb{P}^{m-1}(\mathbb{C})$. Therefore, we prove our Main Theorem for $\tilde{q}$ instead of q.

Step 2: It follows from hypothesis (1.1) that

$$(\sum_{j=1}^{q}(1 - \frac{k}{m_j})) - 2N + k - 1 > \frac{(2N - k + 1)k}{2} > 0 \tag{3.9}$$

holds and by (3.8) this implies in particular

$$q > 2N - k + 1 \ge N + 1 \ge k + 1\,.$$

By Theorem 9, we have

$$(q - 2N + k - 1)\theta = \sum_{j=1}^{q} \omega(j) - k - 1\,; \theta \ge \omega(j) > 0 \text{ and } \theta \ge \frac{k+1}{2N - k + 1}. \tag{3.10}$$

So, using (3.10), we get

$$\begin{aligned} 2\Big((\sum_{j=1}^{q}\omega(j)(1 - \frac{k}{m_j})) - k - 1\Big) &= \frac{2((\sum_{j=1}^{q}\omega(j)) - k - 1)\theta}{\theta} - 2\sum_{j=1}^{q}\frac{k\omega(j)\theta}{\theta m_j} \\ &= 2(q - 2N + k - 1)\theta - 2\sum_{j=1}^{q}\frac{k\omega(j)\theta}{\theta m_j} \\ &\ge 2(q - 2N + k - 1)\theta - 2\sum_{j=1}^{q}\frac{k\theta}{m_j} \\ &= 2\theta\Big((\sum_{j=1}^{q}(1 - \frac{k}{m_j})) - 2N + k - 1\Big) \end{aligned}$$

$$\geq 2\frac{(k+1)\Big((\sum_{j=1}^{q}(1-\dfrac{k}{m_j}))-2N+k-1\Big)}{2N-k+1}.$$

Thus, we now can conclude with (3.9) that

$$2\Big((\sum_{j=1}^{q}\omega(j)(1-\frac{k}{m_j}))-k-1\Big)>k(k+1)$$

$$\Rightarrow (\sum_{j=1}^{q}\omega(j)(1-\frac{k}{m_j}))-k-1-\frac{k(k+1)}{2}>0. \tag{3.11}$$

By (3.11), we can choose a number $\epsilon(>0)\in\mathbb{Q}$ such that

$$\frac{\sum_{j=1}^{q}\omega(j)(1-\frac{k}{m_j})-(k+1)-\frac{k(k+1)}{2}}{\tau_{k+1}}>\epsilon>$$

$$>\frac{\sum_{j=1}^{q}\omega(j)(1-\frac{k}{m_j})-(k+1)-\frac{k(k+1)}{2}}{\frac{1}{q}+\tau_{k+1}}.$$

So

$$h:=(\sum_{j=1}^{q}\omega(j)(1-\frac{k}{m_j}))-(k+1)-\epsilon\sigma_{k+1}>\frac{k(k+1)}{2}+\epsilon\tau_k \tag{3.12}$$

and

$$\frac{\epsilon}{q}>(\sum_{j=1}^{q}\omega(j)(1-\frac{k}{m_j}))-(k+1)-\frac{k(k+1)}{2}-\epsilon\tau_{k+1}. \tag{3.13}$$

We now consider the number

$$\rho:=\frac{1}{h}\Big(\frac{k(k+1)}{2}+\epsilon\tau_k\Big)=\frac{1}{h}\Big(\sigma_k+\epsilon\tau_k\Big). \tag{3.14}$$

Then, by (3.12), we have

$$0<\rho<1. \tag{3.15}$$

Set

$$\rho^*:=\frac{1}{(1-\rho)h}=\frac{1}{(\sum_{j=1}^{q}\omega(j)(1-\frac{k}{m_j}))-(k+1)-\frac{k(k+1)}{2}-\epsilon\tau_{k+1}}. \tag{3.16}$$

Using (3.13) we get

$$\frac{\epsilon\rho^*}{q}>1. \tag{3.17}$$

Now, we put $A=M\setminus K$ and

$$A_1=\{z\in M\setminus K:\psi(G)_{jp}(z)\neq 0 \text{ for all } j=1,\cdots,q \text{ and } p=0,\cdots,k\}.$$

We define a new pseudo metric

$$d\tau^2 = \left(\frac{\Pi_{j=1}^q |G_z(H_j)|^{\omega(j)(1-\frac{k}{m_j})}}{|((G_z)_k)_z|^{1+\epsilon}\Pi_{p=0}^{k-1}\Pi_{j=1}^q |(\psi(G_z)_{jp})_z|^{\epsilon/q}}\right)^{2\rho^*} |dz|^2 \tag{3.18}$$

on A_1. We note that by the transformation formulas (3.2) to (3.5) for a local holomorphic coordinate ξ we have

$$\begin{aligned}&\left(\frac{\Pi_{j=1}^q |G_z(H_j)|^{\omega(j)(1-\frac{k}{m_j})}}{|((G_z)_k)_z|^{1+\epsilon}\Pi_{p=0}^{k-1}\Pi_{j=1}^q |(\psi(G_z)_{jp})_z|^{\epsilon/q}}\right)^{2\rho^*} |dz|^2 \\ &= \left(\frac{\Pi_{j=1}^q |G_\xi(H_j)|^{\omega(j)(1-\frac{k}{m_j})}}{|((G_\xi)_k)_\xi|^{1+\epsilon}\Pi_{p=0}^{k-1}\Pi_{j=1}^q |(\psi(G_\xi)_{jp})_\xi|^{\epsilon/q}}\right)^{2\rho^*} |d\xi|^2\end{aligned} \tag{3.19}$$

so the pseudo metric $d\tau$ is in fact defined independently of the choice of the coordinate.

Next we observe that for any point $z \in A$, we have

$$(\nu_{G_k} - \sum_{j=1}^q \omega(j)\nu_{G(H_j)}(1-\frac{k}{m_j}))(z) \geq 0\,. \tag{3.20}$$

In fact, put $\phi := \frac{|G_k|}{\prod_{j=1}^q |G(H_j)|^{\omega(j)}}$. Observing that by (3.8) for all $j = 1, \cdots, q$ and all $z \in A$ we have either $\nu_{G(H_j)}(z) = 0$ or $\nu_{G(H_j)}(z) \geq m_j > k$, we get

$$\frac{k}{m_j}\nu_{G(H_j)} \geq \min\{\nu_{G(H_j)}, k\}\,.$$

So by Proposition 10 we have

$$\begin{aligned}&\nu_{G_k} - \sum_{j=1}^q \omega(j)\nu_{G(H_j)}(1-\frac{k}{m_j}) \\ &= \nu_\phi + \sum_{j=1}^q \omega(j)\frac{k}{m_j}\nu_{G(H_j)} \\ &\geq \nu_\phi + \sum_{j=1}^q \omega(j)\min\{\nu_{G(H_j)}, k\} \geq 0\,.\end{aligned}$$

Now it is easy to see that $d\tau$ is continuous and nowhere vanishing on A_1. Indeed, for $z_0 \in A_1$ with $\Pi_{j=1}^q G(H_j)(z_0) \neq 0$, $d\tau$ is continuous and not vanishing at z_0. Now assume that there exists $z_0 \in A_1$ such that $G(H_i)(z_0) = 0$ for some i. But by (3.20) and (3.8) we then get that $\nu_{G_k}(z_0) > 0$ which contradicts to $z_0 \in A_1$.

It is easy to see that $d\tau$ is flat. It can be smoothly extended over K. Thus, we have a metric, still call it $d\tau$, on

$$A_1' = A_1 \cup K.$$

Note that $d\tau$ is flat outside the compact set K. The key point is to prove that A_1' is complete in that metric.

Step 3: We proceed by contradiction. If A_1' isn't complete, there is a divergent curve $\gamma(t)$ on A_1' with finite length. We may assume that there is a positive distance d between curve γ and the compact K. Therefore

$\gamma : [0,1) \to A_1$ and γ divergent on A_1', with finite length. It implies that from the point of view of M, there are two cases: either $\gamma(t)$ tends to a point z_0 with

$$\Pi_{p=0}^{k}\Pi_{j=1}^{q}|\psi(G)_{jp}|(z_0) = 0.$$

($\gamma(t)$ tends to the boundary of A_1' as $t \to 1$) or else $\gamma(t)$ tends to the boundary of M as $t \to 1$.

For the former case, then using (3.20) we get

$$\begin{aligned}\nu_{d\tau}(z_0) &= -\Big((\nu_{G_k}(z_0) - \sum_{j=1}^{q}\omega(j)\nu_{G(H_j)}(z_0)(1-\frac{k}{m_j})) + (\epsilon\nu_{G_k}(z_0) \\ &\quad + \frac{\epsilon}{q}\sum_{j=1}^{q}\sum_{p=0}^{k-1}\nu_{\psi(G)_{jp}}(z_0))\Big)\rho^* \\ &\le -\epsilon\rho^*\nu_{G_k}(z_0) - \frac{\epsilon\rho^*}{q}\sum_{j=1}^{q}\sum_{p=0}^{k-1}\nu_{\psi(G)_{jp}}(z_0) \le -\frac{\epsilon\rho^*}{q}.\end{aligned}$$

Thus we can find a positive constant C such that

$$|d\tau| \ge \frac{C}{|z-z_0|^{\frac{\epsilon\rho^*}{q}}}|dz|$$

in a neighborhood of z_0 and then, combining with (3.17), we thus have

$$\int_0^1 d\tau = \infty$$

contradicting the finite length of γ. Therefore the last case occur, that is $\gamma(t)$ tends to the boundary of M as $t \to 1$.

Step 4: Choose t_0 such that

$$\int_{t_0}^{1} d\tau < d/3.$$

We consider a small disk Δ with center at $\gamma(t_0)$. Since $d\tau$ is flat, by Lemma 14, Δ is isometric to an ordinary disk in the plane. Let $\Phi : \{|w| < \eta\} \to \Delta$ be this isometry. Extend Φ, as a local isometry into A_1, to the largest disk $\{|w| < R\} = \Delta_R$ possible. Then $R \le d/3$. Hence, the image under Φ be bounded away from K by distance at least $2d/3$. The reason that Φ cannot be extended to a larger disk is that the image goes to the outside boundary A_1' (it cannot go to points of A_1' with $\Pi_{p=0}^{k}\Pi_{j=1}^{q}|\psi(G)_{jp}|(z_0) = 0$ since we have shown already to be infinitely far away in the metric with respect to these points). More precisely, by again Lemma 14, there exists a point w_0 with $|w_0| = R$ so that $\Phi(\overline{0, w_0}) = \Gamma_0$ is a divergent curve on M.

Our goal is to show that Γ_0 has finite length in the original ds^2 on M, contradicting the completeness of the M.

Step 5: Since we want to use Lemma 11 to finish up step 2, for the rest of the proof of step 2 we consider $G_z = ((g_0)_z, \dots, (g_k)_z)$ as a *fixed globally defined reduced representation of* g by means of the global coordinate z of $A \supset A_1$. (We remark that then we loose of course the invariance of $d\tau^2$ under coordinate changes (3.19), but since z is a global coordinate this will be no problem and we will not need this invariance for the application of Lemma 11.) If again $\Phi : \{w : |w| < R\} \to A_1$ is our maximal local isometry, it is in particular holomorphic and locally biholomorphic. So $f := g \circ \Phi : \{w : |w| < R\} \to \mathbb{P}^k(\mathbb{C})$ is a linearly non-degenerate holomorphic map with fixed global reduced representation

$$F := G_z \circ \Phi = ((g_0)_z \circ \Phi, \cdots, (g_k)_z \circ \Phi) = (f_0, \cdots, f_k)\,.$$

Since Φ is locally biholomorphic, the metric on Δ_R induced from ds^2 (cf. (3.1)) through Φ is given by

$$\Phi^* ds^2 = 2|G_z \circ \Phi|^2 |\Phi^* dz|^2 = 2|F|^2 |\frac{dz}{dw}|^2 |dw|^2\,. \tag{3.21}$$

On the other hand, Φ is locally isometric, so we have

$$|dw| = |\Phi^* d\tau| = \left(\frac{\Pi_{j=1}^q |G_z(H_j) \circ \Phi|^{\omega(j)(1-\frac{k}{m_j})}}{|((G_z)_k)_z \circ \Phi|^{1+\epsilon} \Pi_{p=0}^{k-1} \Pi_{j=1}^q |(\psi(G_z)_{jp})_z \circ \Phi|^{\epsilon/q}}\right)^{\rho^*} |\frac{dz}{dw}||dw|\,.$$

By (3.6) and (3.7) we have

$$((G_z)_k)_z \circ \Phi = ((G_z \circ \Phi)_k)_w (\frac{dw}{dz})^{\sigma_k} = (F_k)_w (\frac{dw}{dz})^{\sigma_k}\,,$$

$$(\psi(G_z)_{jp})_z \circ \Phi = (\psi(G_z \circ \Phi)_{jp})_w \cdot (\frac{dw}{dz})^{\sigma_p} = (\psi(F)_{jp})_w \cdot (\frac{dw}{dz})^{\sigma_p}\,,\ (0 \le p \le k)\,.$$

Hence, by definition of ρ in (3.14), we have

$$\begin{aligned} |\frac{dw}{dz}| &= \left(\frac{\Pi_{j=1}^q |G_z(H_j) \circ \Phi|^{\omega(j)(1-\frac{k}{m_j})}}{|((G_z)_k)_z \circ \Phi|^{1+\epsilon} \Pi_{p=0}^{k-1} \Pi_{j=1}^q |(\psi(G_z)_{jp})_z \circ \Phi|^{\epsilon/q}}\right)^{\rho^*} \\ &= \left(\frac{\Pi_{j=1}^q |F(H_j)|^{\omega(j)(1-\frac{k}{m_j})}}{|(F_k)_w|^{1+\epsilon} \Pi_{p=0}^{k-1} \Pi_{j=1}^q |(\psi(F)_{jp})_w|^{\epsilon/q}}\right)^{\rho^*} \frac{1}{|\frac{dw}{dz}|^{h\rho\rho^*}}. \end{aligned}$$

So by the definition of ρ^* in (3.16), we get

$$\begin{aligned} |\frac{dz}{dw}| &= \left(\frac{|(F_k)_w|^{1+\epsilon} \Pi_{p=0}^{k-1} \Pi_{j=1}^q |(\psi(F)_{jp})_w|^{\epsilon/q}}{\Pi_{j=1}^q |F(H_j)|^{\omega(j)(1-\frac{k}{m_j})}}\right)^{\frac{\rho^*}{1+h\rho\rho^*}} \\ &= \left(\frac{|(F_k)_w|^{1+\epsilon} \Pi_{p=0}^{k-1} \Pi_{j=1}^q |(\psi(F)_{jp})_w|^{\epsilon/q}}{\Pi_{j=1}^q |F(H_j)|^{\omega(j)(1-\frac{k}{m_j})}}\right)^{\frac{1}{h}}\,. \end{aligned}$$

Moreover, $|(\psi(F)_{jp})_w| \le |(F_p)_w(H_j)|$ by the definitions, so we obtain

$$|\frac{dz}{dw}| \le \left(\frac{|(F_k)_w|^{1+\epsilon} \Pi_{p=0}^{k-1} \Pi_{j=1}^q |(F_p)_w(H_j)|^{\epsilon/q}}{\Pi_{j=1}^q |F(H_j)|^{\omega(j)(1-\frac{k}{m_j})}}\right)^{\frac{1}{h}}\,. \tag{3.22}$$

By (3.21) and (3.22), we have

$$\Phi^* ds \leqslant \sqrt{2}|F| \left(\frac{|(F_k)_w|^{1+\epsilon} \Pi_{p=0}^{k-1} \Pi_{j=1}^q |(F_p)_w(H_j)|^{\epsilon/q}}{\Pi_{j=1}^q |F(H_j)|^{\omega(j)(1-\frac{k}{m_j})}}\right)^{\frac{1}{h}} |dw|.$$

By (3.10) and (3.12) all the conditions of Lemma 11 are satisfied. So we obtain by Lemma 11:

$$\Phi^* ds \leqslant C(\frac{2R}{R^2 - |w|^2})^\rho |dw| \,.$$

Since by (3.15) we have $0 < \rho < 1$, it then follows that

$$d_{\Gamma_0} \leqslant \int_{\Gamma_0} ds = \int_{\overline{0,w_0}} \Phi^* ds \leqslant C \cdot \int_0^R (\frac{2R}{R^2 - |w|^2})^\rho |dw| < +\infty,$$

where d_{Γ_0} denotes the length of the divergent curve Γ_0 in M, contradicting the assumption of completeness of M. Thus, we conclude that A_1' is complete.

Step 6: Since the metric on A_1' is flat outside of a compact set K, by a theorem of Huber [6, Theorem 13, p. 61] the fact that A_1' has finite total curvature implies that A_1' is finitely connected. This means that there is a compact subregion of A_1' whose complement is the union of a finite number of doubly-connected regions. Thus, we can first conclude that $\Pi_{p=0}^k \Pi_{j=1}^q |\psi(G)_{jp}|(z)$ can have only a finite number of zeros, and second, that the original surface M is finitely connected. Furthermore, by Osserman [11, Theorem 2.1] each annular ends of A_1', hence of M, is conformally equivalent to a punctured disk. Thus, the Riemann surface M must be conformally equivalent to a compact Riemann surface $\overline{M}$ with a finite number of points removed. In a neighborhood of each of those points the Gauss map G be ramified over H_j with multiplicity at least m_j such that

$$\sum_{j=1}^q (1 - \frac{k}{m_j}) > (k+1)(N - \frac{k}{2}) + (N+1) > 2N - k + 1.$$

By a generalized Picard theorem (Lemma 13), the Gauss map G is not essential at those points. Therefore G can be extended to a holomorphic map from $\overline{M}$ to $\mathbb{P}^k(\mathbb{C})$. If the homology class represented by the image of $G : \overline{M} \to \mathbb{P}^k(\mathbb{C})$ is m times the fundamental homology class of $\mathbb{P}^k(\mathbb{C})$, then we have

$$\iint K dA = -2\pi m$$

as the total curvature of M. This proves the Theorem 1. □

4. The proof of Theorem 2

Proof. For convenience of the reader, we first recall some notations on the Gauss map of minimal surfaces in $\mathbb{R}^3$. Let $x = (x_1, x_2, x_3) : M \to \mathbb{R}^3$ be a non-flat complete minimal surface and $g : M \to \mathbb{P}^1(\mathbb{C})$ its Gauss map. Let z be a local holomorphic coordinate. Set $\phi_i := \partial x_i / \partial z$ $(i = 1, 2, 3)$ and $\phi := \phi_1 - \sqrt{-1}\phi_2$. Then, the (classical) Gauss map $g : M \to \mathbb{P}^1(\mathbb{C})$ is given by

$$g = \frac{\phi_3}{\phi_1 - \sqrt{-1}\phi_2},$$

and the metric on M induced from $\mathbb{R}^3$ is given by

$$ds^2 = |\phi|^2 (1 + |g|^2)^2 |dz|^2 \text{ (see Fujimoto [4]).}$$

We remark that although the ϕ_i, $(i = 1, 2, 3)$ and ϕ depend on z, g and ds^2 do not. Next we take a reduced representation $g = (g_0 : g_1)$ on M and set $||g|| = (|g_0|^2 + |g_1|^2)^{1/2}$. Then we can rewrite

$$ds^2 = |h|^2||g||^4|dz|^2\,, \tag{4.1}$$

where $h := \phi/g_0^2$. In particular, h is a holomorphic map without zeros. We remark that h depends on z, however, the reduced representation $g = (g_0 : g_1)$ is globally defined on M and independent of z. Finally we observe that by the assumption that M is not flat, g is not constant.

Now the proof of Theorem 2 will be completely analogue to the proof of Theorem 1.

Step 1: For each a^j $(1 \le j \le q)$ be distinct points in $\mathbb{P}^1(\mathbb{C})$, we may assume $a^j = (a_0^j : a_1^j)$ with $|a_0^j|^2 + |a_1^j|^2 = 1$ $(1 \le j \le q)$. We set $G_j := a_0^j g_1 - a_1^j g_0$ $(1 \le j \le q)$ for the reduced representation $g = (g_0 : g_1)$ of the Gauss map. By the same argument in the step 1 of the proof of Theorem 1, we also can assume that $m_j \ge 2$ for all $j = 1, \cdots, q$.

Step 2: It follows from the hypothesis of theorem

$$\sum_{j=1}^{q} (1 - \frac{1}{m_j}) > 4$$

that we can take δ with

$$\frac{q - 4 - \sum_{j=1}^{q} \frac{1}{m_j}}{q} > \delta > \frac{q - 4 - \sum_{j=1}^{q} \frac{1}{m_j}}{q+2},$$

and set $p = 2/(q - 2 - \sum_{j=1}^{q} \frac{1}{m_j} - q\delta)$. Then

$$0 < p < 1,\ \frac{p}{1-p} > \frac{\delta p}{1-p} > 1\,. \tag{4.2}$$

For convenience, we will use again some notations as in the proof of Theorem 1.

Put $A = M \setminus K$ and

$$A_1 = \{z \in M \setminus K : W(g_0, g_1)(z) \ne 0 \text{ for all } j = 1, \cdots, q\}.$$

We define a new metric

$$d\tau^2 = |h|^{\frac{2}{1-p}} \left(\frac{\Pi_{j=1}^{q} |G_j|^{1 - \frac{1}{m_j} - \delta}}{|W(g_0, g_1)|} \right)^{\frac{2p}{1-p}} |dz|^2$$

on A_1 (where again $G_j := a_0^j g_1 - a_1^j g_0$ and h is defined with respect to the coordinate z on A_1 and $W(g_0, g_1) = W_z(g_0, g_1)$).

First we observe that $d\tau$ is continuous and nowhere vanishing on A_1. Indeed, h is without zeros on A_1 and for each $z_0 \in A_1$ with $G_j(z_0) \ne 0$ for all $j = 1, \cdots, q$, $d\tau$ is continuous at z_0.

Now, suppose there exists a point $z_0 \in A_1$ with $G_j(z_0) = 0$ for some j. Then $G_i(z_0) \ne 0$ for all $i \ne j$ and $\nu_{G_j}(z_0) \ge m_j \ge 2$. Changing the indices if necessary, we may assume that $g_0(z_0) \ne 0$, so also $a_0^j \ne 0$. So, we get

$$\nu_{W(g_0,g_1)}(z_0) = \nu_{\frac{(a_0^j \frac{g_1}{g_0} - a_1^j)'}{a_0^j}}(z_0) = \nu_{\frac{(G_j/g_0)'}{a_0^j}}(z_0) = \nu_{G_j}(z_0) - 1 > 0. \tag{4.3}$$

This is in contradiction with $z_0 \in A_1$. Thus, $d\tau$ is continuous and nowhere vanishing on A_1. By Proposition 7 a) and the dependence of h on z and the independence of the G_j of z, we also easily see that $d\tau$ is independent of the choice of the coordinate z.

It is easy to see that $d\tau$ is flat. It can be smoothly extended over K. Thus, we have a metric, still call it $d\tau$, on

$$A_1' = A_1 \cup K.$$

Note that $d\tau$ is flat outside the compact set K. The key point is to prove that A_1' is complete in that metric.

Step 3: We proceed by contradiction. If A_1' isn't complete, there is a divergent curve $\gamma(t)$ on A_1' with finite length. We may assume that there is a positive distance d between curve γ and the compact K. Therefore $\gamma : [0, 1) \to A_1$ and γ divergent on A_1', with finite length. It implies that from the point of view of M, there are two cases: either $\gamma(t)$ tends to a point z_0 with

$$W(g_0, g_1)(z_0) = 0$$

($\gamma(t)$ tends to the boundary of A_1' as $t \to 1$) or else $\gamma(t)$ tends to the boundary of M as $t \to 1$. For the former case, if $G_j(z_0) = 0$ for some $j \in \{1, \cdots, q\}$ then we have $G_i(z_0) \neq 0$ for all $i \neq j$ and $\nu_{G_j}(z_0) \geq m_j$. By the same argument as in (4.3) we get that

$$\nu_{W(g_0,g_1)}(z_0) = \nu_{G_j}(z_0) - 1.$$

Thus, since $m_j \geq 2$ we have

$$\begin{aligned}\nu_{d\tau}(z_0) &= \frac{p}{1-p}((1 - \frac{1}{m_j} - \delta)\nu_{G_j}(z_0) - \nu_{W(g_0,g_1)}(z_0))\\ &= \frac{p}{1-p}(1 - (\frac{1}{m_j} + \delta)\nu_{G_j}(z_0)) \leq \frac{p}{1-p}(1 - (\frac{1}{m_j} + \delta)m_j)\\ &\leq -\frac{2\delta p}{1-p}.\end{aligned}$$

If $G_j(z_0) \neq 0$ for all $1 \leq j \leq q$, it is easily to see that $\nu_{d\tau}(z_0) \leq -\frac{p}{1-p}$. So, since $0 < \delta < 1$, we can find a positive constant C such that

$$|d\tau| \geq \frac{C}{|z - z_0|^{\delta p/(1-p)}}|dz|$$

in a neighborhood of z_0. Combining with (4.2), we thus have

$$\int_0^1 d\tau = \infty$$

contradicting the finite length of γ. Therefore the last case occur, that is $\gamma(t)$ tends to the boundary of M as $t \to 1$.

Step 4: By the analogue arguments as in the step 4 of the proof of Theorem 1, that we get the local isometric Φ such that $\Phi(\overline{0, w_0}) = \Gamma_0$ is a divergent curve on M. We also show that Γ_0 has finite length in the original ds^2 on M, contradicting the completeness of the M.

Step 5: The map $\Phi(w)$ is locally biholomorphic, and the metric on Δ_R induced from ds^2 through Φ is given by

$$\Phi^* ds^2 = |h \circ \Phi|^2 ||g \circ \Phi||^4 |\frac{dz}{dw}|^2 |dw|^2 . \tag{4.4}$$

On the other hand, Φ is isometric, so we have

$$|dw| = |d\tau| = \left(\frac{|h|\Pi_{j=1}^{q}|G_j|^{(1-\frac{1}{m_j}-\delta)p}}{|W(g_0,g_1)|^p}\right)^{\frac{1}{1-p}}|dz|$$
$$\Rightarrow |\frac{dw}{dz}|^{1-p} = \frac{|h|\Pi_{j=1}^{q}|G_j|^{(1-\frac{1}{m_j}-\delta)p}}{|W(g_0,g_1)|^p}.$$

Set $f := g(\Phi)$, $f_0 := g_0(\Phi)$, $f_1 := g_1(\Phi)$, $F_j := G_j(\Phi)$. Since

$$W_w(f_0,f_1) = (W_z(g_0,g_1)\circ\Phi)\frac{dz}{dw},$$

we obtain

$$|\frac{dz}{dw}| = \frac{|W(f_0,f_1)|^p}{|h(\Phi)|\Pi_{j=1}^{q}|F_j|^{(1-\frac{1}{m_j}-\delta)p}} \tag{4.5}$$

By (4.4) and (4.5) and by definition of p, therefore, we get

$$\Phi^*ds^2 = \left(\frac{||f||^2|W(f_0,f_1)|^p}{\Pi_{j=1}^{q}|F_j|^{(1-\frac{1}{m_j}-\delta)p}}\right)^2|dw|^2$$
$$= \left(\frac{||f||^{q-2-\sum_{j=1}^{q}(\frac{1}{m_j}-1)-q\delta}|W(f_0,f_1)|}{\Pi_{j=1}^{q}|F_j|^{1-\frac{1}{m_j}-\delta}}\right)^{2p}|dw|^2.$$

Using the Lemma 12, we obtain

$$\Phi^*ds^2 \leqslant C^{2p}.(\frac{2R}{R^2-|w|^2})^{2p}|dw|^2.$$

Since $0 < p < 1$, it then follows that

$$d_{\Gamma_0} \leqslant \int_{\Gamma_0} ds = \int_{\overline{0,w_0}} \Phi^*ds \leqslant C^p.\int_0^R (\frac{2R}{R^2-|w|^2})^p|dw| < +\infty,$$

where d_{Γ_0} denotes the length of the divergent curve Γ_0 in M, contradicting the assumption of completeness of M. Thus, we conclude that A'_1 is complete.

Step 6: We argue similarly to step 6 of the proof of Theorem 1, we completed the Theorem 2. □

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