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## СОДЕРЖАНИЕ

**УДК 531**

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# CALCULATING LINEAR VIBRATION OF MAXWELL SYSTEM USING NEWMARK METHOD

**Abstract**

This paper presents the numerical method for dynamic calculation of third order systems. A numerical algorithm is developed on base of the well-known Newmark integration method to calculate dynamic response of third order systems. Then, we apply this method to calculate linear vibration of Maxwell viscoelastic system.

**Keywords**

Viscoelastic, third order, dynamic, Newmark, linear vibration.

## 1. Introduction

In 1959 Newmark presented a family of single-step integration methods for the solution of structural dynamic problems [1, 2]. During the past time Newmark's method has been applied to the dynamic analysis of many practical engineering structures [3]. It has been modified and improved by many other researchers such as Wilson, Hilber, Hughes and Taylor… However, these methods are only used for the system of second order equations.

Many vibration problems in engineering lead the system of differential equations of third order. In this paper we present the establishing and using Newmark integration method for calculating vibration of third order system.

## 2. The Newmark method for the third order systems

The Newmark method is a single-step integration formula. The state vector of the system at a time $t_{n+1} = t_n + h$ is deduced from the already-known state vector at time $t_n$ through a Taylor expansion of the displacements, velocities and accelerations

$$\boldsymbol{f}\left(t_n+h\right)=\boldsymbol{f}\left(t_n\right)+h\,\dot{\boldsymbol{f}}\left(t_n\right)+\frac{h^2}{2!}\ddot{\boldsymbol{f}}\left(t_n\right)+\ldots+\frac{h^s}{s!}\boldsymbol{f}^{(s)}\left(t_n\right)+\boldsymbol{R}_s, \tag{1}$$

where $\boldsymbol{R}_s$ is the remainder of the development to the order *s*

$$\boldsymbol{R}_s=\frac{1}{s!}\int_{t_n}^{t_n+h}\boldsymbol{f}^{(s+1)}(\tau)\left[t_n+h-\tau\right]^s d\tau. \tag{2}$$

Relation (1) allows us to compute the accelerations, velocities and displacements of a system at time $t_{n+1}$

$$\ddot{\boldsymbol{q}}_{n+1}=\ddot{\boldsymbol{q}}_n+\int_{t_n}^{t_{n+1}}\dddot{\boldsymbol{q}}(\tau)\,d\tau, \tag{3}$$

$$\dot{\boldsymbol{q}}_{n+1}=\dot{\boldsymbol{q}}_n+h\ddot{\boldsymbol{q}}_n+\int_{t_n}^{t_{n+1}}\left(t_{n+1}-\tau\right)\dddot{\boldsymbol{q}}(\tau)\,d\tau, \tag{4}$$

$$\boldsymbol{q}_{n+1}=\boldsymbol{q}_n+h\dot{\boldsymbol{q}}_n+\frac{h^2}{2}\ddot{\boldsymbol{q}}_n+\frac{1}{2}\int_{t_n}^{t_{n+1}}\left(t_{n+1}-\tau\right)^2\dddot{\boldsymbol{q}}(\tau)\,d\tau. \tag{5}$$

Let us express $\dddot{\boldsymbol{q}}(\tau)$ in the time interval $\left[t_n,t_{n+1}\right]$ as a function of $\dddot{\boldsymbol{q}}_n$, $\dddot{\boldsymbol{q}}_{n+1}$ at the interval limits

$$\begin{aligned}\dddot{\boldsymbol{q}}_n&=\dddot{\boldsymbol{q}}(\tau)+\boldsymbol{q}^{(4)}(\tau)\left(t_n-\tau\right)+\boldsymbol{q}^{(5)}(\tau)\frac{\left(t_n-\tau\right)^2}{2}+\ldots\\ \dddot{\boldsymbol{q}}_{n+1}&=\dddot{\boldsymbol{q}}(\tau)+\boldsymbol{q}^{(4)}(\tau)\left(t_{n+1}-\tau\right)+\boldsymbol{q}^{(5)}(\tau)\frac{\left(t_{n+1}-\tau\right)^2}{2}+\ldots\end{aligned} \tag{6}$$

By multiplying the first equation of (6) by $\left(1-\alpha\right)$, the second equation by $\alpha$ and adding two equations then, we obtain

$$\dddot{\boldsymbol{q}}(\tau)=\left(1-\alpha\right)\dddot{\boldsymbol{q}}_n+\alpha\,\dddot{\boldsymbol{q}}_{n+1}+\boldsymbol{q}^{(4)}(\tau)\left[\tau-\alpha h-t_n\right]+\mathrm{O}\left(h^2\boldsymbol{q}^{(5)}\right). \tag{7}$$

Likewise, multiplying equations (6) by $\left(1-2\gamma\right),2\gamma$ and by $\left(1-6\beta\right),6\beta$ yields

$$\dddot{\boldsymbol{q}}(\tau)=\left(1-2\gamma\right)\dddot{\boldsymbol{q}}_n+2\gamma\dddot{\boldsymbol{q}}_{n+1}+\boldsymbol{q}^{(4)}(\tau)\left[\tau-2\gamma h-t_n\right]+\mathrm{O}\left(h^2\boldsymbol{q}^{(5)}\right). \tag{8}$$

$$\dddot{\boldsymbol{q}}(\tau)=\left(1-6\beta\right)\dddot{\boldsymbol{q}}_n+6\beta\dddot{\boldsymbol{q}}_{n+1}+\boldsymbol{q}^{(4)}(\tau)\left[\tau-6\beta h-t_n\right]+\mathrm{O}\left(h^2\boldsymbol{q}^{(5)}\right). \tag{9}$$

Hence, by substituting (7), (8) and (9) in the integral terms of (3), (4) and (5), we obtain the quadrature formulas

$$\int_{t_n}^{t_{n+1}} \ddot{\boldsymbol{q}}(\tau)\,d\tau = (1-\alpha)h\ddot{\boldsymbol{q}}_n + \alpha h\ddot{\boldsymbol{q}}_{n+1} + \boldsymbol{r}_n, \tag{10}$$

$$\int_{t_n}^{t_{n+1}} (t_{n+1}-\tau)\ddot{\boldsymbol{q}}(\tau)\,d\tau = \left(\frac{1}{2}-\gamma\right)h^2\ddot{\boldsymbol{q}}_n + \gamma h^2\ddot{\boldsymbol{q}}_{n+1} + \boldsymbol{r}'_n, \tag{11}$$

$$\frac{1}{2}\int_{t_n}^{t_{n+1}} (t_{n+1}-\tau)^2\ddot{\boldsymbol{q}}(\tau)\,d\tau = \left(\frac{1}{6}-\beta\right)h^3\ddot{\boldsymbol{q}}_n + \beta h^3\ddot{\boldsymbol{q}}_{n+1} + \boldsymbol{r}''_n, \tag{12}$$

The corresponding error measure

$$\begin{aligned}
\boldsymbol{r}_n &= \left(\alpha-\frac{1}{2}\right)h^2\boldsymbol{q}^{(4)}(\tilde{\tau}) + \mathrm{O}\left(h^3\boldsymbol{q}^{(5)}\right),\\
\boldsymbol{r}'_n &= \left(\gamma-\frac{1}{6}\right)h^3\boldsymbol{q}^{(4)}(\tilde{\tau}) + \mathrm{O}\left(h^4\boldsymbol{q}^{(5)}\right), \qquad t_n < \tilde{\tau} < t_{n+1}\\
\boldsymbol{r}''_n &= \left(\beta-\frac{1}{24}\right)h^4\boldsymbol{q}^{(4)}(\tilde{\tau}) + \mathrm{O}\left(h^5\boldsymbol{q}^{(5)}\right).
\end{aligned} \tag{13}$$

The constants $\alpha,\gamma$ and $\beta$ are parameters associated with the quadrate scheme. Choosing values $\alpha=\frac{1}{2},\gamma=\frac{1}{6},\beta=\frac{1}{24}$ leads to linear interpolation of $\ddot{\boldsymbol{q}}(\tau)$

$$\ddot{\boldsymbol{q}}(\tau) = \ddot{\boldsymbol{q}}_n + (\tau-t_n)\frac{\ddot{\boldsymbol{q}}_{n+1}-\ddot{\boldsymbol{q}}_n}{h},$$

If we choose $\alpha=\frac{1}{2},\gamma=\frac{1}{4},\beta=\frac{1}{12}$, we obtain the average value of $\ddot{\boldsymbol{q}}(\tau)$ over the time interval $[t_n, t_{n+1}]$

$$\ddot{\boldsymbol{q}}(\tau) = \frac{\ddot{\boldsymbol{q}}_n + \ddot{\boldsymbol{q}}_{n+1}}{2}.$$

By substituting integrals (10), (11) and (12) into equations (3), (4) and (5), we get the approximation formulas of displacements, velocities and accelerations of system at time $t_{n+1}$

$$\ddot{\boldsymbol{q}}_{n+1} = \ddot{\boldsymbol{q}}_n + (1-\alpha)h\ddot{\boldsymbol{q}}_n + \alpha h\ddot{\boldsymbol{q}}_{n+1}, \tag{14}$$

$$\dot{\boldsymbol{q}}_{n+1} = \dot{\boldsymbol{q}}_n + h\ddot{\boldsymbol{q}}_n + \left(\frac{1}{2}-\gamma\right)h^2\ddot{\boldsymbol{q}}_n + \gamma h^2\ddot{\boldsymbol{q}}_{n+1}, \tag{15}$$

$$\boldsymbol{q}_{n+1} = \boldsymbol{q}_n + h\dot{\boldsymbol{q}}_n + \frac{h^2}{2}\ddot{\boldsymbol{q}}_n + \left(\frac{1}{6}-\beta\right)h^3\ddot{\boldsymbol{q}}_n + \beta h^3\ddot{\boldsymbol{q}}_{n+1}. \tag{16}$$

Thus, we have established the approximation formulas (14), (15), (16) to approach solving the system of third order differential equations.

Let us then assume that the equations of dynamics

$$\boldsymbol{M}\dddot{\boldsymbol{q}} + \boldsymbol{B}\ddot{\boldsymbol{q}} + \boldsymbol{C}\dot{\boldsymbol{q}} + \boldsymbol{K}\boldsymbol{q} = \boldsymbol{f}(t), \tag{17}$$

are linear, i.e., that matrices $\boldsymbol{M}, \boldsymbol{B}, \boldsymbol{C}$ and $\boldsymbol{K}$ are independent of $\boldsymbol{q}$, and let us introduce the numerical scheme (14), (15) and (16) in the equations of motion at time $t_{n+1}$ so as to compute $\dddot{\boldsymbol{q}}_{n+1}$

$$\begin{aligned}\left[\boldsymbol{M} + \alpha h\boldsymbol{B} + \gamma h^2\boldsymbol{C} + \beta h^3\boldsymbol{K}\right]\dddot{\boldsymbol{q}}_{n+1} &= \boldsymbol{f}_{n+1} - \boldsymbol{B}\left[\ddot{\boldsymbol{q}}_n + (1-\alpha)h\dddot{\boldsymbol{q}}_n\right] \\ -\boldsymbol{C}\left[\dot{\boldsymbol{q}}_n + h\ddot{\boldsymbol{q}}_n + \left(\frac{1}{2}-\gamma\right)h^2\dddot{\boldsymbol{q}}_n\right] &- \boldsymbol{K}\left[\boldsymbol{q}_n + h\dot{\boldsymbol{q}}_n + \frac{h^2}{2}\ddot{\boldsymbol{q}}_n + \left(\frac{1}{6}-\beta\right)h^3\dddot{\boldsymbol{q}}_n\right].\end{aligned} \tag{18}$$

By solving a system of linear equations (18) we obtain $\dddot{\boldsymbol{q}}_{n+1}$. Then, by using Newmark formulas (14), (15) and (16) we get accelerations, velocities and displacements $\ddot{\boldsymbol{q}}_{n+1}, \dot{\boldsymbol{q}}_{n+1}$ and $\boldsymbol{q}_{n+1}$. We determine the initial conditions of $\dddot{\boldsymbol{q}}(t_0)$ from the given values of $\boldsymbol{q}(t_0)$, $\dot{\boldsymbol{q}}(t_0)$ and $\ddot{\boldsymbol{q}}(t_0)$

$$\dddot{\boldsymbol{q}}(t_0) = \boldsymbol{M}^{-1}\left[\boldsymbol{f}(t_0) - \boldsymbol{B}\ddot{\boldsymbol{q}}(t_0) - \boldsymbol{C}\dot{\boldsymbol{q}}(t_0) - \boldsymbol{K}\boldsymbol{q}(t_0)\right]. \tag{19}$$

Let us assume that the non-linear dynamic equations of third order systems have the following form

$$\boldsymbol{M}(\boldsymbol{q})\dddot{\boldsymbol{q}} + \boldsymbol{k}(t, \boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}}) = \boldsymbol{f}(t, \boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}}), \tag{20}$$

We have $\dddot{\boldsymbol{q}}_{n+1}$ from equation (16)

$$\dddot{\boldsymbol{q}}_{n+1} = \frac{1}{\beta h^3}(\boldsymbol{q}_{n+1} - \boldsymbol{q}_n) - \frac{1}{\beta h^2}\dot{\boldsymbol{q}}_n - \frac{1}{2\beta h}\ddot{\boldsymbol{q}}_n - \left(\frac{1}{6\beta}-1\right)\dddot{\boldsymbol{q}}_n, \tag{21}$$

By substituting (21) into equations (14) and (15), we obtain

$$\dot{\boldsymbol{q}}_{n+1} = \frac{\gamma}{\beta h}(\boldsymbol{q}_{n+1} - \boldsymbol{q}_n) + \left(1-\frac{\gamma}{\beta}\right)\dot{\boldsymbol{q}}_n + \left(1-\frac{\gamma}{2\beta}\right)h\ddot{\boldsymbol{q}}_n + \left(\frac{1}{2}-\frac{\gamma}{6\beta}\right)h^2\dddot{\boldsymbol{q}}_n, \tag{22}$$

$$\ddot{\boldsymbol{q}}_{n+1} = \frac{\alpha}{\beta h^2}(\boldsymbol{q}_{n+1} - \boldsymbol{q}_n) - \frac{\alpha}{\beta h}\dot{\boldsymbol{q}}_n + \left(1-\frac{\alpha}{2\beta}\right)\ddot{\boldsymbol{q}}_n + \left(1-\frac{\alpha}{6\beta}\right)h\dddot{\boldsymbol{q}}_n. \tag{23}$$

We realize that $\dddot{\boldsymbol{q}}_{n+1}, \ddot{\boldsymbol{q}}_{n+1}, \dot{\boldsymbol{q}}_{n+1}$ are represented by $\boldsymbol{q}_{n+1}$ and the known values of

$\boldsymbol{q}_n, \dot{\boldsymbol{q}}_n, \ddot{\boldsymbol{q}}_n, \dddot{\boldsymbol{q}}_n$. By substituting $\dddot{\boldsymbol{q}}_{n+1}, \ddot{\boldsymbol{q}}_{n+1}, \dot{\boldsymbol{q}}_{n+1}$ into (20), we obtain the system of non-linear algebraic equations with unknown $\boldsymbol{q}_{n+1}$. We have values of $\boldsymbol{q}_{n+1}$ through the iteration method Newton. Then, from equations (21), (22) and (23) we determine values of $\dot{\boldsymbol{q}}_{n+1}, \ddot{\boldsymbol{q}}_{n+1}$ and $\dddot{\boldsymbol{q}}_{n+1}$ with the initial conditions of $\dddot{\boldsymbol{q}}(t_0)$ derived from the equations of dynamics (20)

$$\dddot{\boldsymbol{q}}_0 = \boldsymbol{M}^{-1}(\boldsymbol{q}_0)\left[\boldsymbol{f}(t_0, \boldsymbol{q}_0, \dot{\boldsymbol{q}}_0, \ddot{\boldsymbol{q}}_0) - \boldsymbol{k}(t_0, \boldsymbol{q}_0, \dot{\boldsymbol{q}}_0, \ddot{\boldsymbol{q}}_0)\right]. \tag{24}$$

## 3. Calculating linear vibrations of maxwell systems

### 3.1. Equation of motion

According Newton's second law of motion, we have motion equation of system

$$m\,\ddot{x}(t) + p(t) = F(t) \tag{25}$$

Where $x(t)$ is displacement of mass $m$, $F(t)$ is external force, $p(t)$ là internal force in viscoelastic materials (Fig.1a, Fig.1b).

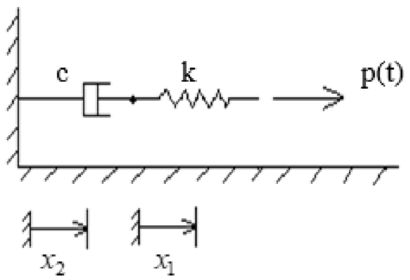
Property of Maxwell model: active forces on elastic element and viscous element are the same, total displacement is sum of ingredient displacements. Thus, we have

$$x = x_1 + x_2 \tag{26}$$

$$p = p_1 = p_2 \tag{27}$$

And $$p_1 = kx_1,\ p_2 = c\dot{x}_2 \tag{28}$$

Derivative of expression (26) $$\dot{x} = \dot{x}_1 + \dot{x}_2 \tag{29}$$



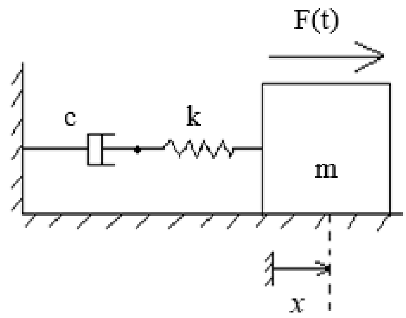




Fig. 1a Fig. 1b

Equation (28) yeilds $\dot{x}_1 = \dfrac{\dot{p}_1}{k},\ \dot{x}_2 = \dfrac{p_2}{c}$. Substituting into (29), we obtain

$$\dot{x} = \frac{\dot{p}_1}{k} + \frac{p_2}{c} \tag{30}$$

From equations (26) and (30) we deduce relation between internal force and displacement

$$\dot{x} = \frac{\dot{p}}{k} + \frac{p}{c} \Rightarrow p + \frac{c}{k}\dot{p} = c\,\dot{x} \tag{31}$$

Substituting equation (25) into equation (31) we obtain the motion equation of Maxwell model

$$\frac{c}{k} m \dddot{x} + m\ddot{x} + c\,\dot{x} = F + \frac{c}{k}\dot{F} \tag{32}$$

### 3.2. Calculating linear vibration

We have the motion equation of Maxwell model at time $t_n$

$$\frac{c}{k} m \dddot{x}_n + m\ddot{x}_n + c\,\dot{x}_n = F_n + \frac{c}{k}\dot{F}_n \tag{33}$$

Applying formulas (22) and (23), we obtain $\dddot{x}_n$

$$\begin{aligned}&\left[a_1 + \alpha h a_2 + \gamma h^2 a_3 + \beta h^3 a_4\right]\dddot{x}_n = f_n - a_2\left[\ddot{x}_{n-1} + (1-\alpha)h\dddot{x}_{n-1}\right] \\ &- a_3\left[\dot{x}_{n-1} + h\ddot{x}_{n-1} + \left(\frac{1}{2} - \gamma\right)h^2\dddot{x}_{n-1}\right] - a_4\left[x_{n-1} + h\dot{x}_{n-1} + \frac{h^2}{2}\ddot{x}_{n-1} + \left(\frac{1}{6} - \beta\right)h^3\dddot{x}_{n-1}\right]\end{aligned} \tag{34}$$

Where $a_1 = \frac{c}{k}m, a_2 = m, a_3 = c, a_4 = 0, f_n = F_n + \frac{c}{k}\dot{F}_n.$

By substituting the above value of $\dddot{x}_n$ into equations (14-16), we get the approximation formulas of displacements, velocities and accelerations of system

$$\begin{aligned}\ddot{x}_n &= \ddot{x}_{n-1} + (1-\alpha)h\dddot{x}_{n-1} + \alpha h\dddot{x}_n, \\ \dot{x}_n &= \dot{x}_{n-1} + h\ddot{x}_{n-1} + \left(\frac{1}{2} - \gamma\right)h^2\dddot{x}_{n-1} + \gamma h^2\dddot{x}_n, \\ x_n &= x_{n-1} + h\dot{x}_{n-1} + \frac{h^2}{2}\ddot{x}_{n-1} + \left(\frac{1}{6} - \beta\right)h^3\dddot{x}_{n-1} + \beta h^3\dddot{x}_n.\end{aligned} \tag{36}$$

With the initial conditions of $x(0), \dot{x}(0), \ddot{x}(0)$, we determine values of $\dddot{x}(0)$:
$\dddot{x}(0) = \frac{k}{cm}\left[F(0) + \frac{c}{k}\dot{F}(0) - m\ddot{x}(0) - c\,\dot{x}(0)\right].$

## 4. Numerical example

We have tried out the algorithm on an example. We have chosen

$$m = 50(\mathrm{kg}), c = 30(\mathrm{Ns/m}), k = 50(\mathrm{N/m}), h = 0.1(\mathrm{s}), \alpha = 0.5, \gamma = 0.25, \beta = 1/12, F = \sin\left(t + \frac{\pi}{4}\right)(\mathrm{N})$$

$$x(0) = 0(\mathrm{m}), \dot{x}(0) = 1(\mathrm{m/s}), \ddot{x}(0) = 0\left(\mathrm{m}/s^2\right) \Rightarrow \dddot{x}(0) = \frac{48}{\sqrt{2}} - 1\left(\mathrm{m/s^2}\right).$$

The differential equation of motion has following form

$$30\dddot{x} + 50\ddot{x} + 30\dot{x} = \sin\left(t + \frac{\pi}{4}\right) + 0.6\cos\left(t + \frac{\pi}{4}\right) \tag{37}$$

The solution of Eq. (37) obtained by the Newmark method is represented in Fig.2.

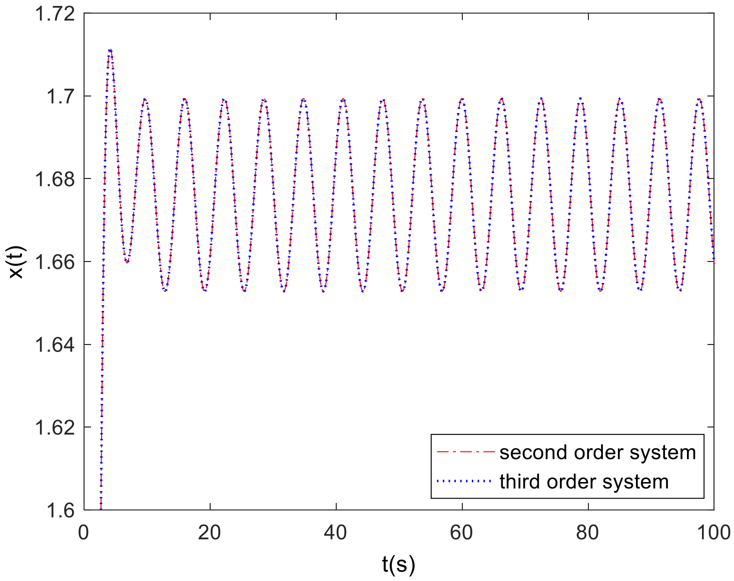



Fig. 2 – Time histories of the displacement in Eq. (37)

## 5. Conclusions

Based on idea of the Newmark integration method, the approximation formulas for the third order dynamic systems are developed in this paper. Then, using the Newmark integration scheme, a numerical algorithm is developed to calculate dynamic response of third order systems. In the example, a good agreement is obtained by the Newmark method between second order system and third order system. The single-step Newmark numerical integration algorithm presented here for Maxwell third order systems is effective and successful. According to this algorithm, a computer program is developed using MATLAB software.

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