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Integer solutions of system of linear inequalities and its aplication to find the extremes of linear and concave functions on the set of integer points of a finite convex polyhedron

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Abstract -While the theory of system of linear inequalities has been well studied, the integer solution of system of linear inequalities has not been properly investigated. One of the main reasons is that this problem is very complicated and it is an NP-hard problem. In this paper we describe a polynomial time algorithm for finding the integer solutions for a system of linear inequalities $Ax \leq b$, where A is (r x n) integer matrix of rank r. From this general case, we come to the algorithm for the paricular case, which is to find a general formula for the integer points in convex cones defined by n linear inequalities. We then propose the procedure of finding integer points in a finite convex polyhedron D, and then develop a method for solving linear integer programming problem over a finite convex polyhedron D. Next, we study the problem of minimizing a concave function on the set of integer points of a finite convex polyhedron D. The problem of finding the minimum of a concave function on the set of integer points of a finite convex polyhedron is a very difficult problem. As far as we know, there is currently no general method to solve this problem. The algorithm proposed here is theoretical, difficult to apply to large problems, except for some problems with special structures.

Keywords: Integer programming, integer point, linear inequalities, convex polyhedron, concave function, global optimization

1. Introduction

The integer programming problem has many practical applications [1,2,3]. There are two traditional exact methods for solving integer programming problems. The first is the cutting plane method. The second way is the branch and bound method [1,2]. But there are also other methods to solve the integer programming problems. For example, the coordinate cutting method [4], or the branch and cut method that combines both branch and bound and cutting plane methods. Since integer programming problem is NP-hard, many problem instances are intractable and so heuristic methods must be used instead [5]. Although the integer programming problem is NP-hard, Lenstra [6] showed that, when the number of variables is fixed, the feasibility integer programming problem can be solved in polynomial time. In practical applications, depending on the structure of the problem, specific solutions are proposed. Algorithms based on the specific structure of the problem are often more effective [1].

In another area, the problem of finding the minimum of a concave function on the set of integer points of a finite convex polyhedron is a very difficult problem. As far as we know, there is currently no general method to solve this problem. The problem of finding the minimum of a concave function on a finite convex polyhedron was first proposed by H. Tuy in the paper [7]. Tuy's cut has since become the classic to open a field called global optimization [8]. However, the field of global optimization on a finite set of integers points is an area where many issues have not been thoroughly investigated.

In present paper we describe a polynomial time algorithm for finding the integer solutions for a system of linear inequalities $Ax \leq b$, where A is (r x n) integer matrix of rank r. Basing on this algorithm we propose a procedure for finding integer points in polytope D, and then develop a method for solving integer programming problems:

$$\max \{ f_0(x) \mid x \in D, x - integer \}$$

And

$$\max \{ \varphi_0(x) \mid x \in D, x - integer \}$$

Where $f_0(x)$ is a linear function, $\varphi_0(x)$ is a concave function and D is a polytope.

2. The integer solutions of a linear system of inequalities

While the theory of system of linear inequalities has been well studied [9], the integer solution of system of linear inequalities has not been properly investigated. This is because the problem of finding the integer solution of any system of linear inequalities is an NP-hard problem. Below, we will refer to the system of linear inequalities whose number of constraints m is not greater than the number of variables n. And with this class of systems of linear inequalities, one can find a general formula for representing integer solutions. Some results on integer solutions of system of linear inequalities can be found in [10].

Let us consider the system of linear inequalities

$$f_i(x) = \sum_{j=1}^{n} a_{ij} x_j \leq b_{i,} i = 1, \ldots, r \quad (1)$$

Where a_{ij},b_i are giver integers and $rank\ (A) = r$.

The following algorithm gives a general formula of integer solutions of the system (1) in polynomial time.

Algorithm

Iteration $k(\ k = 1, \dots., r)$.

Step 0. Define

$$A^* = [a_1^* a_2^* \dots a_{n+r+1}^*] = \begin{bmatrix} AI - b \\ E \end{bmatrix}$$

Where I and E are the identity matrix of orders respectively *r* and $(n + r + 1), -b = (-b_1, \dots, -b_r)^T$

Step 1. Denote the column $j_1 (j_1 \le n - k + 1)$ with

$$\left|a_{1_{j1}}^*\right| = \max_{1 \le j \le n-k+1} |a_{1j}^*|$$

As the operand column.

Step 2. Select the column $j_2 (j_2 \le n - k + 1)$ with

$$\left|a_{1_{j2}}^*\right| = \max_{1 \le j \le n-k+1, j \ne j_1} |a_{1j}^*|$$

And call it the operator column. If no operator column exists, go to step 4.

Step 3. Add an integer multiple of the operator column to the operand column so that the module of the first element of the resulting column is strictly less than the module of the first element of the operator column.

Return to step 1.

Step 4. Compute

$$a_{j1}^* = a_{j1}^* + \lambda_{j1} a_n^*$$

Where a_{j1}^* is the column j_1 of the matrix A^* and λ_{j1} so that.

$$\left|a_{1j}^* + \lambda_{j1} a_{1,n+1}^*\right| = 1$$

Denote the column j_1 as the operator column.

Step 5. Select any column $j(n-k+2 \leq j \leq n, j = n+r-k+2)$ with nonzero element a_{1j}^* as the operand column. If no operand column exists, go to step 7.

Step 6. Add an integer multiple of the operator column to the operand column so that the first element of the resulting column is zero. Return to step 5.

Step 7. Compute

$$a_{n+1}^* = a_{n+1}^* - sgn(a_{1j1}^*)a_{j1}^*$$

Step 8. Expel the first row and the column j_1. If k = r, stop. Otherwise, go to next iteration k=k+1

The set of the integer solution of the system (1) can be represent in the following form.

$$x_i = \sum_{j=1}^{n} a_{ij}^* t_j + a_{i,n+1}^*, i = 1, \dots, n, \quad (2)$$

$$t_j - integer, j = 1, \dots, n, \quad (3)$$

$$t_{n-r+1} \geq -a_{n+1,n+1}^* \,/\, a_{r+1,n-r+1}^*, \quad (4)$$

$$t_j \geq -\left(1/a_{r+j,j}^*\right)\left(\sum_{l=n-r+1}^{j-1} a_{r+j,l}^* t_l + a_{r+j,n+1}^*\right), j = n-r+2, \dots, n$$

3. Generating all integer points of a polytope

Denote by D a feasible set of the following system

$$f_i(x) \leq b_i, i = 1, \dots m. \quad (5)$$

Suppose that D is bounded. Then $rank\ (f_1, \dots, f_m) = n < m$. Without loss of generality, we assume that $(f_1, \dots, f_n)$ are linear independent. Let the above algorithm be applied to the system.

$$f_i(x) \leq b_{i,} i = 1, \dots, n.$$

Then any integer solution of the system (5) is contained in the following set

$$x_i = \sum_{j=1}^{n} a_{ij}^* t_j + a_{i,n+1}^*, i = 1, \dots n, \quad (6)$$

$$t_j - integer, j = 1, \dots n, \quad (7)$$

$$t_1 \geq -{a_{n+1,n+1}^*}/{a_{n+1,1}^*} \quad (8)$$

$$t_j \geq -\left(1/a_{n+j,j}^*\right)\left(\sum_{l=1}^{j-1} a_{n+j,l}^* t_l + a_{n+j,n+1}^*\right), j = 2, \dots, n.$$

Substituting (6) into $f_k(x) \leq b_k, k = n+1, \dots, m$ yields

$$f_k^1(t) = \sum_{j=1}^{n} a_{kj}^1 t_j \leq b_k^1, k = n+1, \dots, m. \tag{9}$$

Therefore, the conditions (6) - (9) define the set of the integer points of the polytope *D*. Let

$$t_1 = \lceil -a_{n+1,n+1}^* / a_{n+1,1}^* \rceil, \tag{10}$$

$$t_j = \lceil -(1 / a_{n+j,j}^*)(\sum_{l=1}^{j-1} a_{n+j,l}^* t_1 + a_{n+j,n+1}^*) \rceil, j = 2, \dots, n,$$

(here $\lceil a \rceil$ represents the smallest integer which is not less than *a)*.

Substituting from (10) in (8), (9) gives $t_{*n} \leq t_n \leq t_n^*$.

Hence, any integer $t_n \in [t_{*n}, t_n^*]$ which together with (10),(6) define an integer point of *D*.

If the interval $[t_{*n}, t_n^*]$ is empty, then in (10) we take

$t_1 = t_1, \dots, \; t_{n-3} = t_{n-3}, t_{n-2} = t_{n-2} + 1$ and repeat this process, e.t.c.

The following theorem gives a condition for stopping the process of increasing t_j.

Theorem 2.1. Suppose that

For $t_1 = t_1^*, \dots, \; t_k = t_k^*$ *there exist* $t_{k+1} = t_{k+1}^1, \dots, t_n = t_n^1$ *such that (6) gives a solution to (5)* (i)

For $t_1 = t_1^*, \dots, \; t_k = t_k^*, t_{k+1} = t_{k+1}^1 + 1$ the formula does not provide a solution to (5) for any $t_{k+2,}, \dots, t_n$. (ii)

Then if $t_1 = t_1^*, \dots, \; t_k = t_k^*, t_{k+1} \geq t_{k+1}^1 + 2$ the formula (6) also does not provide a solution to (5) for any $t_{k+2,}, \dots, t_n$

Similarly, we can find the integer points of the nonconvex sat D $\cap g(x) \geq 0$, where $g(x)$is a convex function.

4. Integer programming problem

Consider the following integer programming problem

$$\text{maximize } f_o(x) \tag{11}$$

$$subject\ to\ f_i(x) \leq b_i, i = 1, \dots, m. \tag{12}$$

$$x - integer, \tag{13}$$

Where $f_0(x)$, $f_1(x)$,…., $f_m(x)$ are linear functions and b_i are giver integers. Suppose that the feasible set *D* of (12) is bounded. Then the algorithm for solving the problem (11) – (13) is constructed as follows.

Algorithm

Step 1. Let x^* be an optimal solution to the problem (11) – (12). If x^*satisfiles (13) then it is also an optimal solution to the problem (11) – (13). Otherwise, go to step 2.

Step 2. (a) Let $I_1 = \left\{ {}^{i}/_{f_i(x^*)} = b_i \right\}$. Using the algorithm in section 1, we find the set of the integer points of the cone

$$f_0(x) \le f_0(x^*),$$

$$f_i(x) \le b_i, i \in I_2 \tag{14}$$

Where $I_2 = \{i \mid i \in I_1\}$ such that $|I_2| = n - 1$ and $rank(f_0(x), f_i(x), i \in I_2) = n$. The order of $f_i(x)$ is defined so that if $i < k$ then $\|f_0 - f_i\| \le \|f_0 - f_k\|$.

(b) Apply the process in section 2 for finding the first integer point x^0 of the system (14) which satisfies the following system $f_i(x) \le b_i, i \in \{1, \dots, m\} \backslash I_2$

If there exists such integer point x^0, then x^0 is an optimal solution of the problem (11) – (13). Otherwise, the problem (11) – (13) has no feasible solutions.

5. Minimize the concave function on a set of integer points

Let us now consider the problem

$$minimize\ \varphi_0(x) \tag{15}$$

subject to (12), (13),

Where φ_0 is a concave function. Suppose that the objective value $\varphi_0(x)$ is integer for every integer point $x \in D$, then the problem (15), (12), (13) can be written as

$$minimize\ t \tag{16}$$

Subject to (12),(13),

$$\varphi_0(x) - t \le 0, \tag{17}$$

$$t - integer \tag{18}$$

Denote $y = (x, t), \varphi_i(y) = f_i(x), g(y) = t - \varphi_0(x), c = (0, \dots., 0, 1)$ we obtain from (16),(12),(13),(18) the following problem.

$$minimize\ (c, y)$$

$$\text{Subject to } \varphi_i(y) \le b_i, i = 1, \dots, m,$$

$$g(y) \ge 0,$$

$$y - integer,$$

Which can be apply by the above algorithm.

We have solved a number of small examples. For some special classes of the problems (11),(12),(13) and (15),(12),(13) more extensive testing is currently being undertaken. The field of global optimization is a difficult field to perform experimental calculations for large problems. Global optimization on the set of integer points is even more difficult. Therefore, the proposed algorithm above is theoretical, applicable to small problems, very difficult to apply to large problems, except for some problems with special structures.

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