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RESEARCH PAPER

NEW FINITE-TIME STABILITY ANALYSIS OF SINGULAR FRACTIONAL DIFFERENTIAL EQUATIONS WITH TIME-VARYING DELAY

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Abstract

The Lyapunov function method is a powerful tool to stability analysis of functional differential equations. However, this method is not effectively applied for fractional differential equations with delay, since the constructing Lyapunov-Krasovskii function and calculating its fractional derivative are still difficult. In this paper, to overcome this difficulty we propose an analytical approach, which is based on the Laplace transform and "inf-sup" method, to study finite-time stability of singular fractional differential equations with interval time-varying delay. Based on the proposed approach, new delay-dependent sufficient conditions such that the system is regular, impulse-free and finite-time stable are developed in terms of a tractable linear matrix inequality and the Mittag-Leffler function. A numerical example is given to illustrate the application of the proposed stability conditions.

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Key Words and Phrases: fractional derivatives; finite-time stability; Laplace transform; Mittag-Leffler function; time delay; linear matrix inequality

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1. **Introduction**

Over the last decades, considerable attention has been paid to stability theory of fractional differential equations (FDEs) (see [8, 9, 15] and the references therein). Stability analysis of singular fractional differential equations (SFDEs) is more complicated than that of ordinary fractional differential equations, because fractional derivatives are nonlocal and have weakly singular kernels and singular systems usually have complicated structure of modes as finite dynamic, non-dynamic modes, impulse modes, which do not appear in the state-space systems. In recent years, various effective methods have been employed to derive stability criteria for SFDEs. The most well-known one is the Lyapunov function method, which was used in [12, 13, 14] by applying the Lyapunov stability theorem extended to FDEs. In addition, Laplace transform and Lambert functions approach ([6, 7]), Gronwall's approach ([10, 23]), Razumikhin approach ([2, 21]) and discrete comparison principle ([22]) are also used to investigate the stability of FDEs. On the other hand, there has been a considerable research interest in study of FDEs with delays. Recently, in [17] the authors proposed some sufficient conditions for finite-time stability of FDEs with constant delay using the inf-sup method. The authors of [19] studied asymptotic behavior of solutions to nonlinear fractional differential equations with constant delay based on the linearization approach. The finite-time stability of SFDEs was studied in [16], but for the system with constant delay. It is worth noting that the methods used in the mentioned paper are not effectively applied for FDEs with time-varying delay. In fact, it is difficult to construct a Lyapunov functional and calculate its fractional derivatives in order to apply fractional Lyapunov stabilty theorems. This is the main reason that there are few results on stability of FDEs with delays. In [5, 25], to overcome the difficulty of calculating the fractional-order derivative the authors attempt to construct an appropriate Lyapunov functional $V(x_t)$ associated with the Riemann-Liouville fractional integral. However, the proof of the main theorem in these papers contains a gap, that is, the positivity of the constructed Lyapunov-Krasovskii functionals can not guarantee the positive definiteness of $V(x_t)$. In [26] the authors proposed a Lyapunov stability theorem for FDEs with delays to derive sufficient stability conditions, unfortunately, the obtained results are also incorrect, since their proof is based on a wrong argument of proving the definite positiveness of the infinite-dimensional Lyapunov functional $V(x_t)$.

Motivated by the above discussion, we consider stability problem of SFDEs with time-varying delay. A central analysis technique is enabled by using Laplace transform approach combining with the "inf-sup" method. The main contribution of this paper is to propose new delay-dependent

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conditions for finite-time stability in the form of an easily verified linear matrix inequality and the Mittag-Leffler function. It should be pointed out that the proposed delay-dependent Lyapunov functional, which is only non-negative definite, makes the derived conditions relatively simple and reliable.

The paper is organized as follows. In Section 2, we provide some preliminaries on fractional derivatives, Laplace transforms, finite-time stability problem and some auxiliary lemmas needed in next section. In Section 3, delay-dependent sufficient conditions for finite-time stability of SFDEs with interval time-varying delay are presented. The effectiveness of the theoretical result is illustrated by a numerical example and its simulation.

Notations: $\mathbb{N}$ denotes the set of all non-negative integers, $\mathbb{R}^+$ denotes the set of all non-negative real numbers; $\mathbb{C}$ denotes the complex space; $\mathbb{R}^n$ denotes the $n-$dimensional Euclidean space with the scalar product $\langle .,. \rangle$; $\mathbb{R}^{n\times r}$ denotes the space of all $(n \times r)-$ matrices; $\lambda(A)$ denotes the set of all eigenvalues of A; $\lambda_{\max}(A) = \max\{Re(\lambda) : \lambda \in \lambda(A)\}$; $\lambda_{\min}(A) = \min\{Re(\lambda) : \lambda \in \lambda(A)\}$; $\|A\|$ denotes the spectral norm defined by $\sqrt{\lambda_{\max}(A^\top A)}$; Matrix A is positive definite $(A > 0)$ if $(Ax, x) > 0$ for all $x \neq 0$. $C([h, 0], \mathbb{R}^n)$ denotes the set of all $\mathbb{R}^n$ c-valued continuously functions on $[h, 0]$; $[a]$ denotes the integer part of a number a; $\Delta V(.)$ denotes the gradient of vector function $V(.)$.

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2. Preliminaries

In this section, we first give some basic concepts of fractional calculus introduced in [8, 15]. The Mittag-Lefller function with two parameters is defined by

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha + \beta)},$$

where $\alpha > 0,\ \beta > 0$, and $z \in \mathbb{C}$. For $\beta = 1$, we denote $E_\alpha(z) := E_{\alpha,1}(z)$.

Lemma 2.1. ([8]). *Given $\alpha > 0$, we have*
(i) $E_\alpha(z) \geq 1,\ z \in \mathbb{R}^+$.
(ii) $E_\alpha(z)$ is increasing on $\mathbb{R}^+$.

For $0 < \alpha < 1, f \in C[0, T]$ the Riemann-Liouville integral $I^\alpha f(t)$ and the Riemann-Liouville derivative $D^\alpha_R f(t)$ are defined respectively by

$$\begin{aligned} I^\alpha f(t) =& \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau, \\ D^\alpha_R f(t) =& \frac{d}{dt} I^{1-\alpha} f(t). \end{aligned}$$

The Caputo fractional derivative $D^\alpha f(t)$ is defined via the Riemann-Liouville derivative as

$$D^\alpha f(t) = D^\alpha_R[f(t) - f(0)].$$

For the Laplace transform $\mathbb{L}[f(t)](s)$ of an integrable function $f(.)$, defined as

$$F(s) = \mathbb{L}[f(t)](s) = \int_0^\infty e^{-st} f(t)dt,$$

we mention the following relations.

Lemma 2.2. ([15]) *Let $f(.) : \mathbb{R}^+ \to \mathbb{R}$ be an integrable function, then we have*:

(i) $\mathbb{L}[D^\alpha f(t)](s) = s^\alpha \mathbb{L}[f(t)](s) - s^{\alpha-1} f(0), \quad \alpha \in (0, 1),$

(ii) For $k \in \mathbb{N}, h > 0, Re(s) > h^{1/\alpha}$,

$$\mathbb{L}[t^{\alpha k+\beta-1} E^{(k)}_{\alpha,\beta}(ht^\alpha)](s) = \frac{k! s^{\alpha-\beta}}{(s^\alpha - h)^{k+1}},$$

(iii) $\mathbb{L}[f * g(t)](s) = \mathbb{L}[f(t)](s) \cdot \mathbb{L}[g(t)](s)$,

where $f(t)$, $g(t)$ are integrable functions on $\mathbb{R}^+$, the convolution of $f(t)$ and $g(t)$ is defined by $f * g(t) = \int_0^t f(t-\tau)g(\tau)d\tau$.

Consider a singular fractional differential equation with interval time-varying delay of the form

$$\begin{cases} ED^\alpha x(t) = Ax(t) + Dx(t - h(t)), \\ x(\theta) = \varphi(\theta), \ \theta \in [-h_2, 0], \end{cases} \tag{2.1}$$

where $\alpha \in (0,\ 1), x(t) \in \mathbb{R}^n$, E is a singular matrix, rank$E = r < n$. $A, D \in \mathbb{R}^{n\times n}$ are given constant matrices, the initial function $\varphi \in C([-h_2, 0], \mathbb{R}^n)$ with the norm $\|\varphi\| = \sup_{t\in[-h_2,0]} \|\varphi(t)\|$; the delay function $h(t)$ is continuous and satisfies the following condition:

$$0 < h_1 \le h(t) \le h_2, \quad t \ge 0.$$

Definition 2.1. ([3, 24]) System (2.1) is said to be (i) regular if for some $s \in \mathbb{C}$ the polynomial $det(s^\alpha E - A)$ is not identically zero; (ii) impulse-free if for some $s \in \mathbb{C}$ the deg(det$(s^\alpha E - A)$) = rank E.

Similar to singular delay systems, system (2.1) may have an impulsive solution, however, the regularity and the absence of impulses of the pair

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(E, A) ensure the existence and uniqueness of an impulse-free solution to the system, which is shown in following lemma.

LEMMA 2.3. ([20]) *Assume that system (2.1) is regular and impulse-free. Then for every initial continuous condition* $\phi(t)$*, system (2.1) has a unique solution on* $[0, +\infty)$.

DEFINITION 2.2. For given positive numbers T, c_1, c_2, system (2.1) is finite-time stable w.r.t. (c_1, c_2, T) if it is regular, impulse-free and every solution $x(t, \varphi)$ of the system satisfies the condition:

$$\sup_{s\in[-h_2,0]} \varphi(s)^\top \varphi(s) \leq c_1 \;\Rightarrow\; x(t,\varphi)^\top x(t,\varphi) \leq c_2, \quad t \in [0, T].$$

Consider system (2.1), where $rank\ E < n$. Then there are two nonsingular matrices M, G such that

$$MEG = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}.$$

Let us set

$$MAG = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad MDG = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix}.$$

Under the state transformation $y(t) = G^{-1}x(t),\ y(t) = (y_1(t),\ y_2(t))$, $y_1(t) \in \mathbb{R}^r,\ y_2(t) \in \mathbb{R}^{n-r}$, the system (2.1) takes the following form

$$\begin{cases} D^\alpha y_1(t) &= A_{11}y_1(t) + A_{12}y_2(t) + D_{11}y_1(t-h(t)) + D_{12}y_2(t-h(t)), \\ 0 &= A_{21}y_1(t) + A_{22}y_2(t) + D_{21}y_1(t-h(t)) + D_{22}y_2(t-h(t)), \\ y(t) &= G^{-1}\varphi(t),\ t \in [-h_2, 0]. \end{cases} \tag{2.2}$$

LEMMA 2.4. ([3, 27]) *System (2.1) is regular and impulse-free if* A_{22} *is invertible.*

LEMMA 2.5. *For any matrices* $A \in \mathbb{R}^{m\times n}, B \in \mathbb{R}^{n\times n}$*, if* $AB + B^T A^T < 0$ *then* A *and* B *are full-row rank and full-column rank, respectively.*

P r o o f. The proof is obvious. □

The following generalized version of Lemma 1 in [4] will be used in the proof of the main result.

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LEMMA 2.6. ([18]) *Let* $\alpha \in (0,1), x(t) \in C([0,+\infty], \mathbb{R}^n)$ *and* $V(.) : \mathbb{R}^n \to \mathbb{R}^+$ *be a convex and differentiable function on* $\mathbb{R}^n$ *such that* $V(0) = 0$. *We have*

$$D^\alpha V(x(t)) \leq \langle \Delta V(x(t)), D^\alpha x(t)) \rangle, \quad t \geq 0.$$

LEMMA 2.7. *Given* $T, h > 0,\ a \geq 1,\ b \geq 0$, *and a non-decreasing function* $G(t) : [-h,\ T] \to \mathbb{R}^+$ *satisfying*

$$G(t) \leq aG(0) + bG(t-h),\ \forall t \in [0, T],$$

we have

$$G(t) \leq G(0) a \sum_{j=0}^{[T/h]+1} b^j, \quad \forall t \in [0,\ T].$$

P r o o f. For each $t \in [0, T]$, there exists $m \in \mathbb{N}$ such that $mh \leq t < (m+1)h$. By induction we have

$$G(t) \leq G(0) \sum_{i=0}^{m} ab^i + b^{m+1} G(t-(m+1)h),$$

for $m \geq 1$, and $G(t) \leq aG(0) + bG(t-(m+1)h)$, for $m = 0$. Since $G(t)$ is non-decreasing on $-h \leq t-(m+1)h < 0$, $G(t-(m+1)h) \leq G(0)$ and $a \geq 1$, we obtain that

$$G(t) \leq \begin{cases} \Big[a + ba + \cdots + b^m a + b^{m+1} a \Big] G(0), & m \geq 1, \\ (a+ba)G(0), & m = 0, \end{cases}$$
$$= a \sum_{j=0}^{m+1} b^j G(0).$$

Besides, $t \leq T$ leads to $m \leq [T/h]$ and hence

$$G(t) \leq a \sum_{j=0}^{[T/h]+1} b^j G(0).$$

The proof is completed. □

3. **Finite-time stability of system (2.1)**

In this section, we present delay-dependent conditions for finite-time stability of system (2.1). The proof is based on the properties of Mittag-Leffler functions, the Laplace form and "inf-sup" method. Before introducing the main result, the following notations of several matrix variables are

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defined for simplicity:

$$W_{11} = PA + A^\top P^\top - h_2 PE, \quad W_{22} = -h_2 PE + RD + D^\top R^\top,$$

$$G^\top PEG = \begin{bmatrix} P_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad p = \frac{\lambda_{\max}(PE)}{\lambda_{\min}(P_1)}, \quad \eta^2 = \lambda_{\max}([G^{-1}]^\top [G^{-1}],$$

$$\beta = \lambda_{\max}(G^\top G), \quad \gamma = \max\{\|A_{22}^{-1} A_{21}\|, \|[A_{22}]^{-1} D_{21}\|, \|[A_{22}]^{-1} D_{22}\|\},$$

$$\gamma_1 = \Big[\max_{i=0,1,2,\ldots,[\frac{T}{h_1}]} 2\sum_{k=0}^{i} \gamma^{k+1} + 2\eta\gamma^{i+2} \Big]^2.$$

Our main result is the following.

THEOREM 3.1. *For given positive numbers T, c_1, c_2, system (2.1) is finite-time stable w.r.t. (c_1, c_2, T) if there exist a non-singular matrix $P \in \mathbb{R}^{n\times n}$, a symmetric positive definite matrix U and a matrix $R \in \mathbb{R}^{n\times n}$, such that the following conditions hold:*

$$PE = E^\top P^\top \geq 0, \tag{3.1}$$

$$\begin{bmatrix} W_{11} & PD & (UA)^\top \\ * & W_{22} & (UD)^\top - R \\ * & * & -2U \end{bmatrix} < 0, \tag{3.2}$$

$$\beta \Big[p(1+\gamma_1) \sum_{j=0}^{[T/h_1]+1} (E_\alpha(h_2 T^\alpha) - 1)^j E_\alpha(h_2 T^\alpha) + \eta^2 \gamma_1 \Big] \leq \frac{c_2}{c_1}. \tag{3.3}$$

P r o o f. The proof is divided into two steps. The first step is to prove the regularity and the impulse-absence of system (2.1). The second step will focus on deriving conditions for finite-time stability by using Laplace transform and "inf-sup" method.

Step 1. The regularity and impulse-absence of the system (2.1). Let us set

$$G^\top = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}, \; G^T S M^{-1} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}, \; G^\top P M^{-1} = \begin{bmatrix} P_1 & P_{12} \\ P_{21} & P_{22} \end{bmatrix}.$$

Using the condition (3.1), we have

$$G^\top PEG = G^\top P M^{-1} MEG = G^\top P M^{-1} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} P_1 & 0 \\ P_{21} & 0 \end{bmatrix} \geq 0,$$

$$G^\top E^\top P^\top G = \begin{bmatrix} P_1^\top & P_{21}^\top \\ 0 & 0 \end{bmatrix} \geq 0,$$

and hence,

$$P_{21} = 0, \; P_1 = P_1^\top \geq 0, \; G^\top PEG = \begin{bmatrix} P_1 & 0 \\ 0 & 0 \end{bmatrix}. \tag{3.4}$$

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Since the matrix P is nonsingular, matrix $G^\top PM^{-1}$ is nonsingular and $G^\top PM^{-1} = \begin{bmatrix} P_1 & P_{12} \\ 0 & P_{22} \end{bmatrix}$, thus from (3.4) it follows that $det(P_1) \neq 0$, $P_1 > 0$. Next, note that LMI (3.2) implies

$$G^\top [PA + A^\top P^\top - h_2 PE]G < 0. \tag{3.5}$$

On the other hand, we formulate the expression of $G^\top PAG, G^\top PEG$ as follows

$$\begin{aligned} G^\top PAG = G^\top PM^{-1}MAG &= \begin{bmatrix} P_1 & P_{12} \\ 0 & P_{22} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \\ &= \begin{bmatrix} P_1 A_{11} + P_{12} A_{21} & P_1 A_{12} + P_{12} A_{22} \\ P_{22} A_{21} & P_{22} A_{22} \end{bmatrix}, \\ G^\top PEG &= \begin{bmatrix} P_1 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

Therefore, taking inequality (3.5) into account we have

$$P_{22} A_{22} + A_{22}^\top P_{22}^\top < 0,$$

which gives, using Lemma 2.5, $det(A_{22}) \neq 0$ and then the system is, by Lemma 2.4, regular and impulse-free.

Step 2. Finite-time stability. Consider the following non-negative convex function:

$$V(x(t)) = x(t)^\top PEx(t).$$

Using Lemma 2.6 and taking the Caputo derivative of $V(.)$ in t along the solution of the system, we have

$$\begin{aligned} &D^\alpha(V(x(t))) \leq 2x(t)^\top PED^\alpha(x(t)) = 2x(t)^\top P\Big(Ax(t) + Dx(t-h(t))\Big) \\ &\leq 2x(t)^\top P\Big(Ax(t) + Dx(t-h(t))\Big) - h_2 x(t-h(t))^\top PEx(t-h(t)) \\ &+ h_2 x(t-h(t))^\top PEx(t-h(t)) - h_2 x(t)^\top PEx(t) + h_2 x(t)^\top PEx(t). \end{aligned} \tag{3.6}$$

Multiplying both sides from the left of system (2.1) by $2(ED^\alpha \dot{x}(t))^\top U$ and $2x(t-h(t))^\top R$, respectively, we obtain that

$$\begin{aligned} 0 &= -2(ED^\alpha \dot{x}(t))^\top UED^\alpha \dot{x}(t) + 2(ED^\alpha \dot{x}(t))^\top U[Ax(t) + Dx(t-h(t))], \\ 0 &= -2x(t-h(t))^\top RED^\alpha \dot{x}(t) + 2x(t-h(t))^\top R[Ax(t) + Dx(t-h(t))]. \end{aligned} \tag{3.7}$$

Hence, we obtain from (3.6), (3.7) that

$$D^\alpha V(\cdot) - h_2 V(\cdot) \leq \xi(t)^\top W\xi(t) + h_2 x(t-h(t))^\top PEx(t-h(t)), \tag{3.8}$$

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where $\xi(t) = [x(t),\, x(t-h(t)),\, ED^{\alpha}\dot{x}(t)]$, and

$$W = \begin{bmatrix} W_{11} & PD & (UA)^{\top} \\ * & W_{22} & (UD)^{\top} - R \\ * & * & -2U \end{bmatrix},$$

$$W_{11} = PA + A^{\top}P^{\top} - h_2PE,\ W_{22} = -h_2PE + RD + D^{\top}R^{\top}.$$

From the conditions (3.2), (3.8) it follows that

$$D^{\alpha}(V(x(t))) - h_2V(x(t)) \le h_2x(t-h(t))^{\top}PEx(t-h(t)).$$

Let us set

$$M(t) = D^{\alpha}(V(x(t))) - h_2V(x(t)), \tag{3.9}$$

then we have

$$M(t) \le h_2x(t-h(t))^{\top}PEx(t-h(t)). \tag{3.10}$$

Applying the Laplace transform to both sides of (3.9), by Lemma 2.2 (i), we have

$$s^{\alpha}\mathbb{V}(s) - V(x(0))s^{\alpha-1} = h_2\mathbb{V}(s) + \mathbb{M}(s),$$

where $\mathbb{V}(s) = \mathbb{L}[V(x(t))](s),\ \mathbb{M}(s) = \mathbb{L}[M(t)](s)$, and hence

$$\mathbb{V}(s) = (s^{\alpha} - h_2)^{-1}(V(x(0))s^{\alpha-1} + \mathbb{M}(s)). \tag{3.11}$$

On the other hand, using (3.10) we have

$$\begin{aligned}\sup_{0\le\tau\le t} M(\tau) \le& h_2 \sup_{0\le\tau\le t} x(\tau-h(\tau))^{\top}PEx(\tau-h(\tau)) \\ \le& h_2 \sup_{-h_2\le\theta\le t-h_1} x(\theta)^{\top}PEx(\theta).\end{aligned}$$

and the following relations hold:

$$(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(h_2(t-\tau)^{\alpha}) \ge 0, \forall t \ge 0,\ \tau \in [0,t],$$

$$\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(h_2(t-\tau)^{\alpha})d\tau = \frac{1}{h_2}[E_{\alpha}(h_2t^{\alpha}) - 1].$$

Therefore, taking the inverse Laplace transform to both sides of equation (3.11) and using Lemma 2.2 (ii)-(iii), we obtain that

$$\begin{aligned}V(x(t)) =& V(0,x(0))E_{\alpha}(h_2t^{\alpha}) + \int_0^t M(\tau)(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(h_2(t-\tau)^{\alpha})d\tau \\ \le& V(x(0))E_{\alpha}(h_2t^{\alpha}) + \sup_{0\le\tau\le t} M(\tau)\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(h_2(t-\tau)^{\alpha})d\tau \\ \le& V(x(0))E_{\alpha}(h_2t^{\alpha}) + \Big(E_{\alpha}(h_2t^{\alpha}) - 1\Big) \sup_{-h_2\le\theta\le t-h_1} x(\theta)^{\top}PEx(\theta),\end{aligned}$$

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and hence

$$x(t)^\top PEx(t) \leq \varphi(0)^\top PE\varphi(0)E_\alpha(h_2t^\alpha) + \Big(E_\alpha(h_2t^\alpha) - 1\Big) \sup_{-h_2\leq\theta\leq t-h_1} x(\theta)^\top PEx(\theta). \qquad (3.12)$$

We now estimate the value $x(\tau)^\top PEx(\tau)$ on $\tau \in [-h_2, t]$. Firstly, note that $E_\alpha(h_2T^\alpha) \geq 1$, using Lemma 2.1 we have for $\tau \in [-h_2, 0]$:

$$x(\tau)^\top PEx(\tau) \leq \sup_{\theta\in[-h_2,0]} \varphi(\theta)^\top PE\varphi(\theta)E_\alpha(h_2T^\alpha).$$

Since $E_\alpha(.)$ is non-decreasing, applying the derived condition (3.12) for $0 \leq \tau \leq t \leq T$, we get

$$\begin{aligned} x(\tau)^\top PEx(\tau) \leq & \varphi(0)^\top PE\varphi(0)E_\alpha(h_2\tau^\alpha) \\ & + \Big(E_\alpha(h_2\tau^\alpha) - 1\Big) \sup_{-h_2\leq\theta\leq\tau-h_1} x(\theta)^\top PEx(\theta) \\ \leq & \sup_{\theta\in[-h_2,0]} \varphi(\theta)^\top PE\varphi(\theta)E_\alpha(h_2T^\alpha) \\ & + \Big(E_\alpha(h_2T^\alpha) - 1\Big) \sup_{-h_2\leq\theta\leq t-h_1} x(\theta)^\top PEx(\theta), \end{aligned}$$

which implies

$$\begin{aligned} \sup_{-h_2\leq\theta\leq t} x(\theta)^\top PEx(\theta) \leq & \sup_{\theta\in[-h_2,0]} \varphi(\theta)^\top PE\varphi(\theta)E_\alpha(h_2T^\alpha) \\ & + \Big(E_\alpha(h_2T^\alpha) - 1\Big) \sup_{-h_2\leq\theta\leq t-h_1} x(\theta)^\top PEx(\theta). \end{aligned}$$

Let us denote

$$G(t) = \sup_{-h_2\leq\theta\leq t} x(\theta)^\top PEx(\theta), \; a = E_\alpha(h_2T^\alpha), \; b = E_\alpha(h_2T^\alpha) - 1,$$

we have

$$G(t) \leq aG(0) + bG(t - h_1), \quad t \in [0, T].$$

From Lemma 2.7 it follows that

$$G(t) \leq G(0)q, \quad t \in [0, T],$$

where $q := \sum_{j=0}^{[T/h_1]+1} (E_\alpha(h_2T^\alpha) - 1)^j E_\alpha(h_2T^\alpha)$. Consequently,

$$x(t)^\top PEx(t) \leq G(t) \leq G(0)q = \sup_{\theta\in[-h_2,0]} \varphi(\theta)^\top PE\varphi(\theta)q, \quad t \in [0, T].$$

Besides, it is easy to see that

$$x(t)^\top PEx(t) = y(t)^\top G(PE)Gy(t) \geq \lambda_{\min}(P_1)\|y_1(t)\|^2,$$

$$\varphi(\theta)^\top PE\varphi(\theta) \leq \lambda_{\max}(PE)\varphi(\theta)^\top\varphi(\theta) \leq \lambda_{\max}(PE)c_1.$$

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Hence, we obtain

$$\|y_1(t)\|^2 \leq \frac{\lambda_{\max}(PE)}{\lambda_{\min}(P_1)} qc_1 = pqc_1, \quad t \in [0, T]. \tag{3.13}$$

Next, we estimate the second state $\|y_2(t\|$ as follows. Consider the second equation of (2.2)

$$y_2(t) = -[A_{22}]^{-1}\Big[A_{21}y_1(t) + D_{21}y_1(t-h(t)) + D_{22}y_2(t-h(t))\Big].$$

Applying estimation (3.13) for $t \in [0, T]$ gives

$$\|y_1(t-h(t))\|^2 \leq \lambda_{\max}([G^{-1}]^\top [G^{-1}])c_1 + \frac{\lambda_{\max}(PE)}{\lambda_{\min}(P_1)} qc_1 = (\eta^2 + pq)c_1,$$

and hence

$$\|y_2(t)\| \leq \zeta\sqrt{c_1} + \gamma\|y_2(t-h(t))\|, \quad t \in [0, T], \tag{3.14}$$

where $\zeta = 2\gamma\sqrt{pq + \eta^2}$. On the other hand, using inequality (3.14) for $t \in [0, h_1]$, we obtain that

$$\|y_2(t)\| \leq (\zeta + \gamma\eta)\sqrt{c_1},$$

because of $t - h_2 \leq t - h(t) \leq t - h_1 \leq 0$, and

$$\|y_2(t-h(t))\|^2 \leq \|y(t-h(t))\|^2 = \lambda_{\max}([G^{-1}]^\top [G^{-1}])c_1 = \eta^2 c_1.$$

By induction, for $t \in [ih_1, (i+1)h_1] \cap [0, T],\ ih_1 \leq T,\ i = 0, 1, ...,$ we have

$$\|y_2(t)\| \leq \zeta[\sum_{k=0}^{i} \gamma^k + \eta\gamma^{i+1}]\sqrt{c_1},$$

and hence,

$$\|y_2(t)\| \leq \sqrt{(\eta^2 + pq)\gamma_1 c_1}, \tag{3.15}$$

where $\gamma_1 = \Big[\max\limits_{i=0,1,2,\dots,[\frac{T}{h_1}]} 2\sum\limits_{k=0}^{i} \gamma^{k+1} + 2\eta\gamma^{i+2}\Big]^2$. Finally, combining conditions (3.13), (3.15) with condition (3.3), we obtain that

$$\begin{aligned}
\|x(t,\phi)\|^2 &= x(t,\phi)^\top x(t,\phi) = y(t)^\top G^\top G y(t) \leq \lambda_{\max}(G^\top G)\|y(t)\|^2 \\
&= \beta(\|y_1(t)\|^2 + \|y_2(t)\|^2) \\
&\leq \beta(pq + (\eta^2 + pq)\gamma_1)c_1 \leq c_2, \quad t \in [0, T].
\end{aligned}$$

The proof of the theorem is completed. □

Remark 3.1. In Theorem 3.1 the conditions (3.1), (3.2) guarantee the regularity and impulse-absence of the system. Note that the condition (3.3) is not an LMI, but it can be reduced into a single strict LMI. Moreover, since the parameters c_1, c_2, do not involve in the conditions (3.1), (3.2), we first determine solutions P, U, R from the conditions and then verify condition (3.3).

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Remark 3.2. In [5, 12, 25, 26], the authors proposed the delay-dependent Lyapunov functional associated with the Riemann-Liouville fractional integral $V(x_t) = I^{\alpha}[x^{\top}(t)PEx(t)] + \int_{t-h}^{t} x^T(s)Qx(s)ds$, where $Q > 0$. However, the constructed Lyapunov functional can not guarantee the positive definiteness of the function $V(x_t)$, since the second integral functional is obviously not greater than $\lambda||x(t)||^2$ for some $\lambda > 0$, and hence the use of the fractional Lyapunov stability theorem was impossible. To avoid the use of the fractional Lyapunov stability theorem, we have constructed delay-independent Lyapunov-like functional $V(x(t)) = x^{\top}(t)PEx(t)$, which is only non-negative definite in order to apply Lemma 2.6. This approach allows us not only to overcome the positive definiteness of the Lyapunov functional, but also to derived delay-dependent sufficient conditions for finite-time stability.

Example 3.1. Consider system (2.1), where $\alpha = 1/2$, $h(t) = 0.09 + 0.01sin^2(t)$, and

$$E = \begin{bmatrix} 10 & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -2 & 1 \\ 1 & -10 \end{bmatrix},$$
$$D = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \; M = G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

We have $h_1 = 0.09$, $h_2 = 0.1$. By using LMI Toolbox in Matlab, the matrix inequalities (3.1), (3.2) are feasible with

$$P = 10^4 \begin{bmatrix} 1.2553 & 0.0000 \\ 0.0000 & 0.5393 \end{bmatrix}, \; U = 10^3 \begin{bmatrix} 7.0204 & -0.1299 \\ -0.1299 & 1.3583 \end{bmatrix},$$
$$R = 10^3 \begin{bmatrix} -3.0006 & -0.7289 \\ -6.8905 & -1.2284 \end{bmatrix}.$$

For $c_1 = 1;\ c_2 = 3,\ T = 10$, calculating

$$PE = E^{\top}P = 10^5 \begin{bmatrix} 1.2553 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\beta = \eta = p = 1;\ \gamma = 0.1,\ \left[T/h_1\right] = 111, \gamma_1 = 0.0123,$$

we can verify the condition (3.3) as

$$\beta\Big[p(1+\gamma_1) \sum_{j=0}^{[T/h_1]+1} (E_{\alpha}(h_2T^{\alpha}) - 1)^j E_{\alpha}(h_2T^{\alpha}) + \eta^2\gamma_1\Big]$$
$$= 2.9449 < \frac{c_2}{c_1}.$$

Hence, by Theorem 3.1 the system (2.1) is finite-time stable w.r.t. $(1, 3, 10)$.

Figure 1 shows the trajectories of $x^\top(t)x(t)$ of the system with the initial condition $\varphi(t) = (0.7,\ 0.7)^\top$, $t \in [-0.1,\ 0]$.

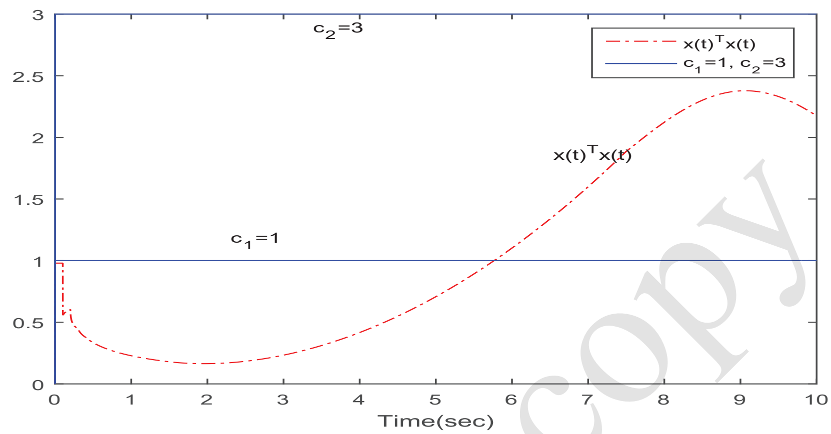

FIGURE 1. State response of $x^\top(t)x(t)$

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4. Conclusion

We have studied the finite-time stability of SFDEs with interval time-varying delay. By proposing an analytical approach based on the Laplace transform and "inf-sup" method, we have derived delay-dependent sufficient conditions for finite-time stability in terms of the Mittag-Leffler function and a tractable matrix inequality. An illustrative example with simulation is given to show the validity and effectiveness of the derived result. Extending the obtained results for the finite-time stability of singular non-autonomous FDEs with time-varying delay is a worthy of attention research subject for future works.

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